PAPER

Which Replacement Is Better at Working Cycles or Number of Failures

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SUMMARY When a unit repeats some works over again and undergoes minimal repairs at failures, it is more practical to replace it preventively at the end of working cycles or at its failure times. In this case, it would be an interesting problem to know which is better to replace the unit at a number of working cycles or at random failures from the point of cost. For this purpose, we give models of the expected cost rates for the following replacement policies: (1) The unit is replaced at a working cycle N and at a failure number K, respectively; (2) Replacement first and last policies with working cycle N and failure number K, respectively; (3) Replacement overtime policies with working cycle N and failure number K, respectively. Optimizations and comparisons of the policies for N and K are made analytically and numerically.

key words: *replacement policies, minimal repair, working cycle, replacement last, replacement overtime*

1. Introduction

It has become an important problem to plan good maintenance policies for a large-scale system such as a plant equipment and an information system with network, as they have been widely used in various environments and their sudden failures might incur great losses or even social confusions. There have been many research works regarding to preventive maintenance policies in theory [1]–[4]; however, the difficulties are which maintenance policies are better from the points of cost, practicality and reliability. Further, if we could rank the maintenance policies as needed, but will this rank change when the maintenance environment changes. In order to answer the above questions, this paper tries to give some comparisons of replacement policies when an operating unit is replaced at working cycles or at random failures.

In general, it would be impossible to do some maintenance during the interval when the unit is operating for works, and it would be better to do maintenance when the work completes or when the unit fails. The theoretical models have shown that maintaining a unit after it completes some works are possible even though they are sometimes costly [5]. The properties of replacement times between two successive failed units were investigated for a system which replaced only at random times [6]. Several random and age replacement models were discussed [7] for an operating unit which repeats some works over again. The maintenance model with age T and number of jobs completed N has been considered [8]. Considering the systems successively executing jobs with random working times, it would be better to conduct the maintenance after the jobs completed. Other maintenance models with random working cycle have been studied extensively [9]–[11]. Furthermore, replacement first and last with two kinds of failures were considered and their optimal policies were discussed and compared [12]. Replacement policies in which a maintainer made the postponed replacement in a delay time due to inspection test were studied [13].

Nakagawa and Zhao considered about first and last policies for replacement and inspections when the policies are triggered by two factors. [14]–[16]. *Replacement first* means that the unit is replaced preventively at time of events such as operating time, number of repairs, working cycles, cumulative damage, etc, whichever occurs first, and *replacement last* means that the unit is replaced preventively at the above events, whichever occurs last. It has been shown that [15] replacement last policies could let the unit operate works as longer as possible while replacement first polices are more easily to save total maintenance cost. Replacement first and last polices are good alternatives when the unit performs one big project and the decision on replacement is based on the termination time of the project [10].

For replacement first and last policies, it is an interesting problem to determine which policies are better from the points of cost, practicality and reliability. The recent work [17] has given some comparative methods for replacement policies when they are performed at continuous or discrete times. However, when the unit can be only replaced preventively at discrete times such as working cycles or at random failures, this paper will answer the questions like how we can formulate the replacement first and last policies, and how we can know which policy is better from the point of cost. For this purpose, the following policies are given: (1) The unit is replaced at the Nth ($N = 1, 2, \dots$) working cycle, and the Kth ($K = 1, 2, \dots$) failure, respectively; (2) the unit is replaced at the Nth working cycle or the Kth failure, whichever occurs first and last. In addition, it is a possible way to delay the replacement policies over a planned time, whose models are called replacement overtime [18], so that the above policies are extended to overtime replacement policies with working cycles N and failures K.

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We formulate the expected cost rates, give analytical discussions and make comparisons to decided which is better for the above replacement policies in each sections. Finally, we give the properties of extended failure rates which are needed for theoretical analysis of optimal policies in Appendix.

2. Assumptions

We give the following assumptions for the models in this paper:

1. The unit repeats some works over again which have random working cycles Y_j $(j = 1, 2, \dots)$. It is assumed that Y_j are independent and identically distributed random variables and have an identical distribution $G(t) \equiv \Pr\{Y_j \le t\}$ with finite mean $1/\theta \equiv \int_0^\infty \overline{G}(t)dt$, where $\overline{G}(t) \equiv 1 - G(t)$. Let $G^{(n)}(t)$ $(n = 1, 2, \dots)$ denote the n-fold Stieltjes convolution of $G(t)$ and $G^{(0)}(t) \equiv 1$ for $t \ge 0$.
2. It is assumed that failures occur in a nonhomogeneous Poisson process with $H(t) \equiv \int_0^t h(u)\mathrm{d}u$. Let $p_j(t)$ denote the probability that the number of failures in $[0, t]$ is j, i.e.,
$$p_j(t) \equiv \frac{H(t)^j}{j!}\mathrm{e}^{-H(t)} \qquad (j = 0, 1, 2, \dots). \tag{1}$$
We assume that the failure rate $h(t)$ increases strictly from $h(0) = 0$ to $h(\infty) = \infty$ for simplicity of discussions.
3. Let $P_K(t) \equiv \sum_{j=K}^\infty p_j(t)$ and $\overline{P}_K(t) \equiv \sum_{j=0}^{K-1} p_j(t)$ $(K = 0, 1, 2, \dots)$, where note that $\sum_{j=0}^{-1} \equiv 0$. Further, we have the following relations for t $(0 < t < \infty)$ and j $(j = 0, 1, 2, \dots)$,
$$P_{j+1}(t) = \int_0^t p_j(u)h(u)\mathrm{d}u,$$
$$\overline{P}_{j+1}(t) = \int_t^\infty p_j(u)h(u)\mathrm{d}u, \qquad \int_0^\infty p_j(t)h(t)\mathrm{d}t = 1,$$
$$\int_0^\infty H(t)\mathrm{d}P_j(t) = \int_0^\infty \overline{P}_j(t)h(t)\mathrm{d}t = \sum_{i=0}^{j-1}\int_0^\infty p_i(t)h(t)\mathrm{d}t = j.$$
4. The probability that some failures occur in $(0, t]$ is given by $F(t) \equiv \sum_{j=1}^\infty p_j(t) = 1 - p_0(t) = 1 - \mathrm{e}^{-H(t)}$ with finite mean μ, and $f(t)$ is a density function of $F(t)$ and $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$. Thus, for given t $(0 \le t < \infty)$, the probability that a failure occurs in $(u, u + \mathrm{d}u]$ is $f(u)\mathrm{d}u/\overline{F}(t)$ for $u > t$.
5. When the failure has occurred, the unit is replaced or undergoes *minimal repair*. The unit after minimal repair has the same failure rate as before rate [3, p. 96].

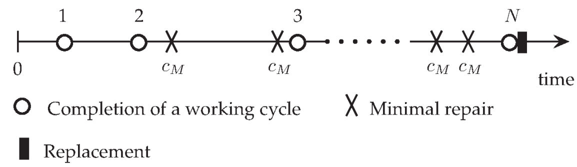

Fig. 1 Replacement at cycle N.

3. Basic Policies

3.1 Replacement at Cycle N

Suppose that the unit is replaced at working cycle N $(N = 1, 2, \dots)$ (see Fig. 1). Then, the expected cost rate is [5, p. 76],
$$C(N) = \frac{c_N + c_M \int_0^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t}{N/\theta} \qquad (N = 1, 2, \dots), \tag{2}$$
where c_N = replacement cost at cycle N and c_M = cost of minimal repair at each failure.

We find optimal N^* to minimize $C(N)$. Forming the inequality $C(N + 1) - C(N) \ge 0$,
$$\int_0^\infty [1 - G^{(N)}(t)][Q_1(N) - h(t)]\mathrm{d}t \ge \frac{c_N}{c_M}, \tag{3}$$
where for $0 < T \le \infty$ and $N = 0, 1, 2, \dots$,
$$Q_1(N; T) \equiv \frac{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]h(t)\mathrm{d}t}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]\mathrm{d}t} \le h(T),$$
$$Q_1(N) \equiv \lim_{T\to\infty} Q_1(N; T) = \theta\int_0^\infty [G^{(N)}(t) - G^{(N+1)}(t)]h(t)\mathrm{d}t.$$

In particular, when $G(t) = 1 - \mathrm{e}^{-\theta t}$,
$$Q_1(N) = \int_0^\infty \frac{\theta(\theta t)^N}{N!}\mathrm{e}^{-\theta t}h(t)\mathrm{d}t,$$
which increases strictly with N to ∞ from Appendix A. Thus, there exists a finite and unique minimum N^* $(1 \le N^* < \infty)$ which satisfies (3), and the resulting cost rate is
$$c_M Q_1(N^* - 1) < C(N^*) \le c_M Q_1(N^*). \tag{4}$$

3.2 Replacement at Failure K

Suppose that the unit is replaced at failure K $(K = 1, 2, \dots)$ (see Fig. 2). Then, the expected cost rate is [3, p. 106]
$$C(K) = \frac{c_K + c_M K}{\int_0^\infty \overline{P}_K(t)\mathrm{d}t} \qquad (K = 1, 2, \dots), \tag{5}$$

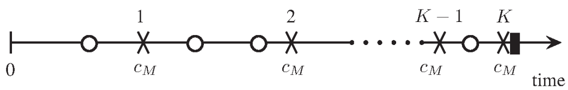

Fig. 2 Replacement at failure K.

where c_K = replacement cost at failure K.

We find optimal K^* to minimize $C(K)$. Forming the inequality $C(K+1) - C(K) \geq 0$,

$$H_1(K)\int_0^\infty \overline{P}_K(t)\mathrm{d}t - K \geq \frac{c_K}{c_M}, \tag{6}$$

where for $0 < T \leq \infty$ and $K = 0, 1, 2, \ldots,$

$$H_1(K;T) \equiv \frac{\int_0^T p_K(t)h(t)\mathrm{d}t}{\int_0^T p_K(t)\mathrm{d}t},$$

$$H_1(K) \equiv \lim_{T\to\infty} H_1(K;T) = \frac{1}{\int_0^\infty p_K(t)\mathrm{d}t},$$

which increases strictly with K to $h(\infty)$ from Appendix B. Thus, because the left-hand side of (6) increases strictly with K to ∞, there exists a finite and unique minimum K^* ($1 \leq K^* < \infty$) which satisfies (6), and the resulting cost rate is

$$c_M H_1(K^*-1) < C(K^*) \leq c_M H_1(K^*). \tag{7}$$

3.3 Numerical Examples

We compute numerically optimal N^* and K^* when $G(t) = 1 - \mathrm{e}^{-t}$ and $H(t) = (\lambda t)^2$, i.e., $h(t) = 2\lambda^2 t$. In this case,

$$Q_1(N) = \int_0^\infty \frac{t^N}{N!}\mathrm{e}^{-t} 2\lambda^2 t\mathrm{d}t = 2\lambda^2(N+1),$$

$$\sum_{j=0}^{N-1}\int_0^\infty \frac{t^j}{j!}\mathrm{e}^{-t}2\lambda^2 t\mathrm{d}t = \lambda^2 N(N+1),$$

and from (3), optimal N^* is given by

$$2\lambda^2 N(N+1) - \lambda^2 N(N+1) = \lambda^2 N(N+1) \geq \frac{c_N}{c_M}, \tag{8}$$

and from (2),

$$\frac{C(N^*)}{c_M} = \frac{c_N/c_M + \lambda^2 N^*(N^*+1)}{N^*}. \tag{9}$$

Further, we can see that

$$\int_0^\infty p_K(t)\mathrm{d}t = \int_0^\infty \frac{(\lambda t)^{2K}}{K!}\mathrm{e}^{-(\lambda t)^2}\mathrm{d}t = \frac{1}{2\lambda}\frac{\Gamma(K+1/2)}{\Gamma(K+1)},$$

$$\sum_{j=0}^{K-1}\int_0^\infty p_j(t)\mathrm{d}t = \frac{1}{\lambda}\frac{\Gamma(K+1/2)}{\Gamma(K)},$$

and from (6), optimal K^* is given by

$$\frac{2\Gamma(K+1/2)/\Gamma(K)}{\Gamma(K+1/2)/\Gamma(K+1)} - K = K \geq \frac{c_K}{c_M}, \tag{10}$$

Table 1 Optimal N^*, K^*, $C(K^*)/c_M$ and $C(N^*)/c_M$ when $G(t) = 1 - \mathrm{e}^{-t}$, $H(t) = (\lambda t)^2$ and $c_N = c_K$.

$\frac{c_N}{c_M}$	K^*	$\lambda = 0.1$			$\lambda = 1$		
		N^*	$\frac{C(K^*)}{c_M}$	$\frac{C(N^*)}{c_M}$	N^*	$\frac{C(K^*)}{c_M}$	$\frac{C(N^*)}{c_M}$
1	1	10	0.226	0.210	1	2.257	3.000
2	2	14	0.301	0.293	1	3.009	4.000
3	3	17	0.361	0.357	2	3.611	4.500
4	4	20	0.413	0.410	2	4.127	5.000
5	5	22	0.459	0.457	2	4.585	5.500
6	6	22	0.500	0.503	2	5.002	6.000
7	7	24	0.539	0.542	3	5.387	6.333
8	8	26	0.575	0.578	3	5.746	6.667
9	9	28	0.608	0.611	3	6.084	7.000
10	10	32	0.640	0.643	3	6.404	7.333

where $\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ for $\alpha > 0$. Thus, if c_K/c_M is an integer then $K^* = c_K/c_M$, and from (5),

$$\frac{C(K^*)}{c_M} = \frac{c_K/c_M + K^*}{\Gamma(K^*+1/2)/[\lambda\Gamma(K^*)]}. \tag{11}$$

Table 1 gives optimal K^*, N^*, $C(K^*)/c_M$ and $C(N^*)/c_M$ for $\lambda = 0.1$, 1, and $c_N/c_M = 1, 2, \ldots 10$. We can see that for $\lambda = 1$, $C(K^*)/c_M < C(N^*)/c_M$, that is, replacement with K^* is better than replacement with N^*. On the other hand, for $\lambda = 0.1$, $C(K^*)/c_M > C(N^*)/c_M$ for $K^* = c_N/c_M \leq 5$, and $C(K^*)/c_M < C(N^*)/c_M$ for $K^* \geq 6$. Optimal N^* decreases with λ. The reason would be that when λ is large, interval times of failures become small and we should replace early to avoid the cost of failures.

4. Replacement First and Last

4.1 Replacement First

The unit is replaced at cycle N ($N = 1, 2, \ldots$) or failure K ($K = 1, 2, \ldots$), whichever occurs first (see Fig. 3). The probability that the unit is replaced at cycle N is $\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t)$, and the probability that it is replaced at failure K is $\int_0^\infty [1 - G^{(N)}]\mathrm{d}P_K(t)$. The mean time to replacement is

$$\int_0^\infty t\,\overline{P}_K(t)\mathrm{d}G^{(N)}(t) + \int_0^\infty t\,[1 - G^{(N)}(t)]\mathrm{d}P_K(t)$$
$$= \int_0^\infty [1 - G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t, \tag{12}$$

and the expected number of failures until replacement is

$$\int_0^\infty H(t)\overline{P}_K(t)\mathrm{d}G^{(N)}(t) + \int_0^\infty H(t)[1 - G^{(N)}(t)]\mathrm{d}P_K(t)$$
$$= \int_0^\infty [1 - G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t. \tag{13}$$

Therefore, the expected cost rate is

$$C_F(N,K) = \frac{c_K - (c_K - c_N)\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t) + c_M\int_0^\infty [1 - G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t}{\int_0^\infty [1 - G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t}. \tag{14}$$

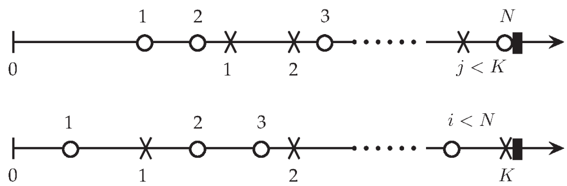

Fig. 3 Replacement first with N and K.

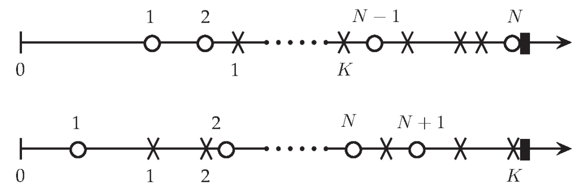

Fig. 4 Replacement last with N and K.

Clearly, $C_F(\infty, K) = C(K)$ in (5) and $C_F(N, \infty) = C(N)$ in (2).

We find optimal N_F^* and K_F^* to minimize $C_F(N, K)$ when $c_K = c_N$ and $G(t) = 1 - \mathrm{e}^{-\theta t}$. Forming the inequality $C_F(N+1, K) - C_F(N, K) \geq 0$,

$$Q_2(N,K)\int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t - \int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t \geq \frac{c_N}{c_M}, \tag{15}$$

where

$$Q_2(N,K) \equiv \frac{\sum_{j=0}^{K-1}\int_0^\infty (\theta t)^N \mathrm{e}^{-\theta t} p_j(t)h(t)\mathrm{d}t}{\sum_{j=0}^{K-1}\int_0^\infty (\theta t)^N \mathrm{e}^{-\theta t} p_j(t)\mathrm{d}t},$$

which increases strictly with N from $Q_2(0, K)$ to $h(\infty)$ and increases strictly with K from $Q_2(N, 1)$ to

$$Q_2(N,\infty) = \int_0^\infty \frac{\theta(\theta t)^N}{N!}\mathrm{e}^{-\theta t}h(t)\mathrm{d}t = Q_1(N)$$

from Appendix A. Thus, because the left-hand side of (13) increases strictly with N to ∞, there exists a finite and unique minimum N_F^* $(1 \leq N_F^* < \infty)$ which satisfies (15), and the resulting cost rate is

$$c_M Q_2(N_F^*-1, K) < C_F(N_F^*, K) \leq c_M Q_2(N_F^*, K). \tag{16}$$

In addition, noting that the left-hand side of (15) goes to that of (3) as $K \to \infty$, N_F^* approaches to N^* given in (3) as $K \to \infty$.

Forming the inequality $C_F(N, K+1) - C_F(N, K) \geq 0$,

$$H_2(K,N)\int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t - \int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t \geq \frac{c_N}{c_M}, \tag{17}$$

where

$$H_2(K,N) \equiv \frac{\sum_{j=0}^{N-1}\int_0^\infty [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t)h(t)\mathrm{d}t}{\sum_{j=0}^{N-1}\int_0^\infty [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t)\mathrm{d}t},$$

which increases strictly with N from $H_2(K, 1)$ to

$$H_2(K,\infty) = \frac{\int_0^\infty p_K(t)h(t)\mathrm{d}t}{\int_0^\infty p_K(t)\mathrm{d}t} = H_1(K),$$

and increases strictly with K from $H_2(0, N)$ to $h(\infty)$ from Appendix B. Thus, because the left-hand side of (17) increases strictly with K to ∞, there exists a finite and unique minimum K_F^* $(1 \leq K_F^* < \infty)$ which satisfies (17), and the resulting cost rate is

$$c_M H_2(K_F^*-1, N) < C_F(N, K_F^*) \leq c_M H_2(K_F^*, N). \tag{18}$$

In addition, noting that the left-hand side of (17) goes to that of (6) as $N \to \infty$, K_F^* approaches to K^* given in (6) as $N \to \infty$.

4.2 Replacement Last

The unit is replaced at cycle N $(N = 0, 1, 2, \ldots)$ or failure K $(K = 0, 1, 2, \ldots)$, whichever occurs last (see Fig. 4). The probability that the unit is replaced at cycle N is $\int_0^\infty P_K(t)\mathrm{d}G^{(N)}(t)$ and the probability that it is replaced at failure K is $\int_0^\infty G^{(N)}(t)\mathrm{d}P_K(t)$. The mean time to replacement is

$$\int_0^\infty tP_K(t)\mathrm{d}G^{(N)}(t) + \int_0^\infty tG^{(N)}(t)\mathrm{d}P_K(t) = \int_0^\infty [1-G^{(N)}(t)P_K(t)]\mathrm{d}t, \tag{19}$$

and the expected number of failures until replacement is

$$\int_0^\infty H(t)P_K(t)\mathrm{d}G^{(N)}(t) + \int_0^\infty H(t)G^{(N)}(t)\mathrm{d}P_K(t) = \int_0^\infty [1-G^{(N)}(t)P_K(t)]h(t)\mathrm{d}t. \tag{20}$$

Therefore, the expected cost rate is

$$C_L(N,K) = \frac{c_K - (c_N - c_K)\int_0^\infty P_K(t)\mathrm{d}G^{(N)}(t) + c_M\int_0^\infty [1-G^{(N)}(t)P_K(t)]h(t)\mathrm{d}t}{\int_0^\infty [1-G^{(N)}(t)P_K(t)]\mathrm{d}t}. \tag{21}$$

Clearly, $C_L(0, K) = C(K)$ in (5) and $C_L(N, 0) = C(N)$ in (2).

We find optimal N_L^* and K_K^* to minimize $C_L(N, K)$ when $c_K = c_N$ and $G(t) = 1 - \mathrm{e}^{-\theta t}$. Forming the inequality $C_L(N+1, K) - C_L(N, K) \geq 0$,

$$\widetilde{Q}_2(N,K)\int_0^\infty [1-G^{(N)}(t)P_K(t)]\mathrm{d}t - \int_0^\infty [1-G^{(N)}(t)P_K(t)]h(t)\mathrm{d}t \geq \frac{c_K}{c_M}, \tag{22}$$

where

$$\widetilde{Q}_2(N,K) \equiv \frac{\sum_{j=K}^{\infty}\int_0^{\infty}(\theta t)^N \mathrm{e}^{-\theta t} p_j(t)h(t)\mathrm{d}t}{\sum_{j=K}^{\infty}\int_0^{\infty}(\theta t)^N \mathrm{e}^{-\theta t} p_j(t)\mathrm{d}t},$$

which increases strictly with N from $\widetilde{Q}_2(0,K)$ to $h(\infty)$ and increases strictly with K from

$$\widetilde{Q}_2(N,0) = \int_0^{\infty}\frac{\theta(\theta t)^N}{N!}\mathrm{e}^{-\theta t}h(t)\mathrm{d}t = Q_1(N)$$

to $h(\infty)$, and $\widetilde{Q}_2(N,K) \geq Q_2(N,K)$ from Appendix C. Thus, because the left-hand side of (22) increases strictly with N to ∞, there exists a finite and unique minimum N_L^* $(0 \leq N_L^* < \infty)$ which satisfies (22), and the resulting cost rate is

$$c_M\widetilde{Q}_2(N_L^*-1,K-1) < C_L(N_L^*,K) \leq c_M\widetilde{Q}_2(N_L^*,K-1). \tag{23}$$

In addition, noting that the left-hand side of (22) agrees with that of (3) when $K = 0$, $N_L^* = N^*$ given in (3) when $K = 0$.

Forming the inequality $C_L(N,K+1) - C_L(N,K) \geq 0$,

$$\widetilde{H}_2(K,N)\int_0^{\infty}[1-G^{(N)}(t)P_K(t)]\mathrm{d}t - \int_0^{\infty}[1-G^{(N)}(t)P_K(t)]h(t)\mathrm{d}t \geq \frac{c_K}{c_M}, \tag{24}$$

where

$$\widetilde{H}_2(K,N) \equiv \frac{\int_0^{\infty}G^{(N)}(t)p_K(t)h(t)\mathrm{d}t}{\int_0^{\infty}G^{(N)}(t)p_k(t)\mathrm{d}t},$$

which increases strictly with K from $\widetilde{H}_2(0,N)$ to $h(\infty)$ from Appendix D. Thus, because the left-hand side of (24) increases strictly with K to ∞, there exists a finite and unique minimum K_L^* $(0 \leq K_L^* < \infty)$ which satisfies (24), and the resulting cost rate is

$$c_M\widetilde{H}_2(K_L^*-1,N) < C_L(N,K_L^*) \leq c_M\widetilde{H}_2(K_L^*,N). \tag{25}$$

In addition, noting that the left-hand side of (24) agrees with that of (6) when $N = 0$, $K_L^* = K^*$ given in (6) when $N = 0$.

We compute numerically optimal (K_F^*, N_F^*) and (K_L^*, N_L^*) when $c_N = c_K$, $G(t) = 1 - \mathrm{e}^{-t}$ and $H(t) = (\lambda t)^2$. Tables 2 and 3 give (K_F^*, N_F^*), $C_F(K_F^*, N_F^*)/c_M$, (K_L^*, N_L^*) and $C_L(K_L^*, N_L^*)/c_M$ for $c_N/c_M = 1, 2, \dots 10$ when $\lambda = 0.1$ and $\lambda = 1$, respectively. We can see from these tables that $C_F(K_F^*, N_F^*)/c_M < C_L(K_L^*, N_L^*)/c_M$.

5. Overtime Policies

5.1 Replacement at First Failure Over Cycle N

Suppose that the unit is replaced at the first failure over working cycle N $(N = 0, 1, 2, \dots)$ (see Fig. 5). Recall that $F(t) = 1 - \mathrm{e}^{-H(t)}$ and the probability that the unit with age t fails in $(u, u + \mathrm{d}u]$ for $u > t$ is $f(u)\mathrm{d}u/\overline{F}(t)$. Thus, the mean time to replacement is

$$\int_0^{\infty}\frac{1}{\overline{F}(t)}\left[\int_t^{\infty}u\,\mathrm{d}F(u)\right]\mathrm{d}G^{(N)}(t) = \frac{N}{\theta} + \int_0^{\infty}\left[\int_t^{\infty}\mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]\mathrm{d}G^{(N)}(t) = \mu + \int_0^{\infty}\left[1-G^{(N)}(t)\right]\left[\int_t^{\infty}\mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]h(t)\mathrm{d}t, \tag{26}$$

and the expected number of failures until replacement is

$$\int_0^{\infty}\frac{1}{\overline{F}(t)}\left[\int_t^{\infty}H(u)\mathrm{d}F(u)\right]\mathrm{d}G^{(N)}(t) = 1 + \int_0^{\infty}[1-G^{(N)}(t)]h(t)\,\mathrm{d}t. \tag{27}$$

Thus, the expected cost rate is

$$C_O(N) = \frac{c_{ON} + c_M\left\{1 + \int_0^{\infty}\left[1-G^{(N)}(t)\right]h(t)\mathrm{d}t\right\}}{\mu + \int_0^{\infty}[1-G^{(N)}(t)]\left[\int_t^{\infty}\mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]h(t)\mathrm{d}t}. \tag{28}$$

where c_{ON} = replacement cost at first failure over cycle N.

We find optimal N_O^* to minimize $C_O(N)$ when $G(t) =$

Table 2 Optimal (K_F^*, N_F^*), $C_F(K_F^*, N_F^*)/c_M$, (K_L^*, N_L^*) and $C_L(K_L^*, N_L^*)/c_M$ when $G(t) = 1 - \mathrm{e}^{-t}$, $H(t) = (\lambda t)^2$, $c_N = c_K$ and $\lambda = 0.1$.

$\frac{c_N}{c_M}$	(K_F^*, N_F^*)	(K_L^*, N_L^*)	$\frac{C_F(K_F^*,N_F^*)}{c_M}$	$\frac{C_L(K_L^*,N_L^*)}{c_M}$
1	(3, 11)	(0, 9)	0.208	0.210
2	(4, 15)	(1, 13)	0.291	0.292
3	(5, 19)	(2, 16)	0.354	0.355
4	(6, 22)	(3, 18)	0.407	0.408
5	(7, 25)	(4, 20)	0.454	0.455
6	(8, 27)	(5, 22)	0.496	0.497
7	(9, 30)	(6, 24)	0.535	0.536
8	(10, 32)	(7, 25)	0.572	0.572
9	(11, 34)	(8, 26)	0.606	0.607
10	(12, 36)	(9, 28)	0.638	0.639

Table 3 Optimal (K_F^*, N_F^*), $C_F(K_F^*, N_F^*)/c_M$, (K_L^*, N_L^*) and $C_L(K_L^*, N_L^*)/c_M$ when $G(t) = 1 - \mathrm{e}^{-t}$, $H(t) = (\lambda t)^2$, $c_N = c_K$ and $\lambda = 1.0$.

$\frac{c_N}{c_M}$	(K_F^*, N_F^*)	(K_L^*, N_L^*)	$\frac{C_F(K_F^*,N_F^*)}{c_M}$	$\frac{C_L(K_L^*,N_L^*)}{c_M}$
1	(2, 3)	(2, 1)	2.221	2.617
2	(3, 4)	(3, 1)	2.995	3.222
3	(4, 5)	(4, 1)	3.604	3.750
4	(5, 5)	(5, 1)	4.123	4.224
5	(6, 6)	(6, 1)	4.583	4.657
6	(7, 7)	(7, 1)	5.001	5.056
7	(8, 7)	(8, 1)	5.386	5.429
8	(9, 8)	(9, 1)	5.745	5.779
9	(10, 8)	(9, 1)	6.083	6.110
10	(11, 9)	(10, 1)	6.404	6.426

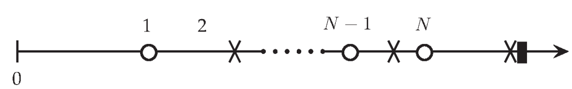

Fig. 5 Replacement at first failure over cycle N.

$1-\mathrm{e}^{-\theta t}$. Forming the inequality $C_O(N+1)-C_O(N)\geq 0$,

$$Q_{O1}(N)\left\{\mu+\int_0^\infty\left[1-G^{(N)}(t)\right]\left[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]h(t)\mathrm{d}t\right\} - \int_0^\infty [1-G^{(N)}(t)]h(t)\mathrm{d}t - 1 \geq \frac{c_{ON}}{c_M}, \tag{29}$$

where

$$Q_{O1}(N)\equiv\frac{\int_0^\infty(\theta t)^N\mathrm{e}^{-\theta t}h(t)\,\mathrm{d}t}{\int_0^\infty(\theta t)^N\mathrm{e}^{-\theta t}h(t)\left[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]\mathrm{d}t},$$

which increases strictly with N to $h(\infty)$ from Appendix E. Thus, because the left-hand side of (29) increases strictly with N to ∞, there exists a finite and unique minimum N_O^* $(0\leq N_O^*<\infty)$ which satisfies (29), and the resulting cost rate is

$$c_M Q_{O1}(N_O^*-1) < C_O(N_O^*) \leq c_M Q_{O1}(N_O^*). \tag{30}$$

5.2 Replacement at First Cycle Over Failure K

Suppose that the unit is replaced at the first working cycle over failure K $(K=0,1,2,\dots)$ (see Fig. 6). The mean time to replacement is

$$\begin{aligned}&\sum_{j=0}^\infty\int_0^\infty\left\{\int_0^t\left[\int_t^\infty y\mathrm{d}G(y-u)\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t)\\ &=\int_0^\infty \overline{P}_K(t)\mathrm{d}t\\ &\quad+\sum_{j=0}^\infty\int_0^\infty\left\{\int_0^t\left[\int_t^\infty \overline{G}(y-u)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t)\\ &=\frac{1}{\theta}\sum_{j=0}^\infty\int_0^\infty G^{(j)}(t)\mathrm{d}P_K(t),\end{aligned} \tag{31}$$

and the expected number of failures until replacement is

$$\begin{aligned}&\sum_{j=0}^\infty\int_0^\infty\left\{\int_0^t\left[\int_t^\infty H(y)\mathrm{d}G(y-u)\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t)\\ &=\sum_{j=0}^\infty\int_0^\infty\left\{\int_0^t\left[\int_t^\infty \overline{G}(y)h(u+y)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t).\end{aligned} \tag{32}$$

Therefore, the expected cost rate is

$$C_O(K)=\frac{c_{OK}+c_M\sum_{j=0}^\infty\int_0^\infty\{\int_0^t[\int_0^\infty \overline{G}(y)h(u+y)\mathrm{d}y]\mathrm{d}G^{(j)}(u)\}\mathrm{d}P_K(t)}{(1/\theta)\sum_{j=0}^\infty\int_0^\infty G^{(j)}(t)\mathrm{d}P_K(t)}. \tag{33}$$

Fig. 6 Replacement at first cycle over failure K.

where c_{OK} = replacement cost at first cycle over failure K. In particular, when $G(t)=1-\mathrm{e}^{-\theta t}$,

$$C_O(K)=\frac{c_{OK}+c_M\{\int_0^\infty \mathrm{e}^{-\theta t}h(t)\mathrm{d}t+\int_0^\infty \overline{P}_K(t)[\int_0^\infty \theta\mathrm{e}^{-\theta u}h(t+u)\mathrm{d}u]\mathrm{d}t\}}{1/\theta+\int_0^\infty \overline{P}_K(t)\mathrm{d}t}. \tag{34}$$

We find optimal K_O^* to minimize $C_O(K)$ in (34). Forming the inequality $C_O(K+1)-C_O(K)\geq 0$,

$$H_{O1}(K)\left[\frac{1}{\theta}+\int_0^\infty \overline{P}_K(t)\mathrm{d}t\right]-\int_0^\infty \mathrm{e}^{-\theta t}h(t)\mathrm{d}t - \int_0^\infty \overline{P}_K(t)\left[\int_0^\infty \theta\mathrm{e}^{-\theta u}h(t+u)\mathrm{d}u\right]\mathrm{d}t\geq\frac{c_{OK}}{c_M}, \tag{35}$$

where for $0<T\leq\infty$,

$$H_{O1}(K,T)\equiv\frac{\int_0^T H(t)^K\mathrm{e}^{-H(t)}\left[\int_0^\infty \theta\mathrm{e}^{-\theta u}h(t+u)\mathrm{d}u\right]\mathrm{d}t}{\int_0^T H(t)^K\mathrm{e}^{-H(t)}\mathrm{d}t},$$

$$\begin{aligned}H_{O1}(K)&\equiv\lim_{T\to\infty}H_{O1}(K,T)\\ &=\frac{\int_0^\infty H(t)^K\mathrm{e}^{-H(t)}\left[\int_0^\infty \theta\mathrm{e}^{-\theta u}h(t+u)\mathrm{d}u\right]\mathrm{d}t}{\int_0^\infty H(t)^K\mathrm{e}^{-H(t)}\mathrm{d}t},\end{aligned}$$

which increases strictly with K to $\int_0^\infty \theta\mathrm{e}^{-\theta t}h(T+t)\mathrm{d}t$ from Appendix F. Thus, because the left-hand side of (35) increases strictly with K to ∞, there exists a finite and unique minimum K_O^* $(0\leq K_O^*<\infty)$ which satisfies (35), and the resulting cost rate is

$$c_M H_{O1}(K_O^*-1) < C_O(K_O^*) \leq c_M H_{O1}(K_O^*). \tag{36}$$

We discuss numerically optimal N_O^*, K_O^* when $c_{ON}=c_{OK}$, $G(t)=1-\mathrm{e}^{-t}$ and $H(t)=(\lambda t)^2$. Tables 4 and 5 give optimal N_O^*, $C_O(N^*)$, K_O^*, $C_O(K_O^*)$ when $\lambda=0.1$ and $\lambda=1$, respectively. We can see from these tables that $C_O(N_O^*)<C_O(K_O^*)$ for $c_{ON}/c_M\leq 5$ and $C_O(N_O^*)\geq C_O(K_O^*)$ for $c_{ON}/c_M\geq 6$ in Table 4, $C_O(N_O^*)<C_O(K_O^*)$ for $c_{ON}/c_M\leq 3$ and $C_O(N_O^*)\geq C_O(K_O^*)$ for $c_{ON}/c_M\geq 4$ in Table 5. This means that c_{ON}/c_M has a threshold level and if c_{ON}/c_M is smaller than this level, i.e., replacement cost is relatively

Table 4 Optimal N_O^*, $C_O(N_O^*)/c_M$, K_O^*, $C_O(K_O^*)/c_M$, when $G(t)=1-\mathrm{e}^{-t}$, $H(t)=(\lambda t)^2$ and $c_{ON}=c_{OK}$, $\lambda=0.1$.

c_{ON}/c_M	N_O^*	$\frac{C_O(N_O^*)}{c_M}$	K_O^*	$\frac{C_O(K_O^*)}{c_M}$
1	6	0.216	1	0.223
2	11	0.294	2	0.300
3	14	0.357	3	0.361
4	17	0.410	4	0.412
5	20	0.457	5	0.458
6	22	0.500	6	0.500
7	24	0.539	7	0.539
8	26	0.576	8	0.575
9	28	0.610	9	0.608
10	30	0.643	10	0.641

Table 5 Optimal N_O^*, $C_O(N_O^*)/c_M$, K_O^*, $C_O(K_O^*)/c_M$, when $G(t) = 1 - \mathrm{e}^{-t}$, $H(t) = (\lambda t)^2$ and $c_{ON} = c_{OK}$, $\lambda = 1$.

c_{ON}/c_M	N_O^*	$\frac{C_O(N_O^*)}{c_M}$	K_O^*	$\frac{C_O(K_O^*)}{c_M}$
1	1	2.702	1	3.060
2	1	3.377	1	3.590
3	1	4.052	1	4.121
4	1	4.728	2	4.576
5	2	5.231	2	5.005
6	2	5.667	3	5.381
7	2	6.103	4	5.743
8	2	6.539	4	6.084
9	3	6.885	5	6.401
10	3	7.198	6	6.707

smaller against minimal repair cost c_M, replacement at first failure over cycle N is better than replacement at first cycle over failure K.

6. Conclusions

We have discussed theoretically and numerically the optimal policies of replacements with cycle N and failure K. In general, it would be more difficult to derive theoretically optimal policies for replacements with discrete variables than those with continuous ones. This paper has given several mathematical techniques of solving optimization problems with discrete variables and these would be useful for maintenances of actual models in practical fields. For example, we can propose the following replacement policies from the results of this paper:

1. If c_N is smaller than c_K and cycle N can be counted more easily than failure K, then cycle N is better than failure K.
2. If c_K is smaller than c_N and failure K can be counted more easily than cycle N, then failure K is better than cycle N.
3. If c_K is smaller than c_N and cycle N can be counted more easily than failure K, then replacement overtime with cycle N is better than failure K.
4. If c_N is smaller than c_K and failure K can be counted more easily than cycle N, then replacement overtime with failure K is better than cycle N.
5. If both costs of c_N and c_K are almost the same and cycle N and failure K can be counted easily, we compute the expected costs $C(N^*)$ and $C(K^*)$, and the expected costs $C(N_F^*, K_F^*)$ and $C(N_L^*, K_L^*)$ numerically, and decide the optimal replacement policy.

The replacement policies proposed in this paper would be applied to the cumulative damage models and data backup models of computer systems by making some suitable modifications [19], [20].

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Appendix A:

For $N = 0, 1, 2, \ldots$, $K = 1, 2, \ldots$ and $0 < T < \infty$,

$$Q_2(N, K; T) \equiv \frac{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} \overline{P}_K(t) h(t) \mathrm{d}t}{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} \overline{P}_K(t) \mathrm{d}t},$$

which increases strictly with N from $Q_2(0,K;T)$ to $h(T)$ and increases strictly with K from $Q_2(N,1;T)$ to

$$Q_2(N;\infty;T)=\frac{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}h(t)\mathrm{d}t}{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\mathrm{d}t}.$$

Proof. Note that

$$Q_2(\infty;K;T)=\lim_{N\to\infty}\frac{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\overline{P}_K(t)h(t)\mathrm{d}t}{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\overline{P}_K(t)\mathrm{d}t}=h(T),$$

$$Q_2(N;\infty;T)=\frac{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}h(t)\mathrm{d}t}{\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\mathrm{d}t}.$$

Denoting

$$L_1(T)\equiv\int_0^T(\theta t)^{N+1}\mathrm{e}^{-\theta t}\overline{P}_K(t)h(t)\mathrm{d}t\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\overline{P}_K(t)\mathrm{d}t-\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\overline{P}_K(t)h(t)\mathrm{d}t\int_0^T(\theta t)^{N+1}\mathrm{e}^{-\theta t}\overline{P}_K(t)\mathrm{d}t,$$

we have $L_1(0)=0$ and

$$L_1'(T)=(\theta T)^N\mathrm{e}^{-\theta T}\overline{P}_K(t)\times\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}\overline{P}_K(t)(\theta T-\theta t)[h(T)-h(t)]\mathrm{d}t>0,$$

which follows that $Q_2(N,K;T)$ increases strictly with N from $Q_2(0,K;T)$ to $h(T)$ for any K and T. Similarly, denoting

$$L_2(T)\equiv\sum_{j=0}^{K}\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}p_j(t)h(t)\mathrm{d}t\sum_{j=0}^{K-1}\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}p_j(t)\mathrm{d}t-\sum_{j=0}^{K-1}\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}p_j(t)h(t)\mathrm{d}t\sum_{j=0}^{K}\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}p_j(t)\mathrm{d}t,$$

we have $L_2(0)=0$ and

$$L_2'(T)=(\theta T)^N\mathrm{e}^{-\theta T}\sum_{j=0}^{K-1}\int_0^T(\theta t)^N\mathrm{e}^{-\theta t}[h(T)-h(t)]\times\frac{\mathrm{e}^{-H(T)-H(t)}}{K!j!}[H(T)H(t)]^j[H(T)^{K-j}-H(t)^{K-j}]\mathrm{d}t>0,$$

which follows that $Q_2(N,K;T)$ increases strictly with K from $Q_2(N,1;T)$ to $Q_2(N;\infty;T)$ for any N and T. Therefore, because T is arbitrary, $Q_2(N,K)\equiv Q_2(N,K;\infty)$ increases strictly with N from $Q_2(0,K)$ to $h(\infty)$ and increases strictly with K from $Q_2(N,1)$ to $Q_1(N)=\theta\int_0^\infty[(\theta t)^N/N!]\mathrm{e}^{-\theta t}h(t)\mathrm{d}t$, and because K is arbitrary, $Q_1(N)$ increases strictly with N to $h(\infty)$.

Appendix B:

For $N=1,2,\ldots$, $K=0,1,2,\ldots$ and $0<T<\infty$,

$$H_2(N,K;T)=\frac{\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)h(t)\mathrm{d}t}{\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)\mathrm{d}t},$$

which increases strictly with N from $H_2(1,K;T)$ to

$$H_2(\infty,K;T)=\frac{\int_0^T p_K(t)h(t)\mathrm{d}t}{\int_0^T p_K(t)\mathrm{d}t},$$

and increases strictly with N from $H_2(N,0;T)$ to $h(T)$.

Proof. Note that

$$H_2(\infty,K;T)=\frac{\int_0^T p_K(t)h(t)\mathrm{d}t}{\int_0^T p_K(t)\mathrm{d}t},$$

$$H_2(N,\infty;T)\equiv\lim_{K\to\infty}\frac{\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)h(t)\mathrm{d}t}{\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)\mathrm{d}t}=h(T).$$

Denoting

$$L_3(T)\equiv\int_0^T[(\theta t)^N/N!]\mathrm{e}^{-\theta t}p_K(t)h(t)\mathrm{d}t\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)\mathrm{d}t-\int_0^T[(\theta t)^N/N!]\mathrm{e}^{-\theta t}p_K(t)\mathrm{d}t\sum_{j=0}^{N-1}\int_0^T[(\theta t)^j/j!]\mathrm{e}^{-\theta t}p_K(t)h(t)\mathrm{d}t,$$

we have $L_3(0)=0$ and

$$L_3'(T)=\sum_{j=0}^{N-1}\int_0^T\mathrm{e}^{-\theta(T+t)}p_K(T)p_K(t)[h(T)-h(t)]\times\frac{(\theta T)^N(\theta t)^j-(\theta T)^j(\theta t)^N}{N!j!}\mathrm{d}t>0,$$

which follows that $H_2(N,K;T)$ increases strictly with N from $H_2(1,K;T)$ to $H_2(\infty,K;T)$ for any K and T. Similarly, denoting

$$L_4(T)\equiv\sum_{j=0}^{N-1}\int_0^T\frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}p_{K+1}(t)h(t)\mathrm{d}t\sum_{j=0}^{N-1}\int_0^T\frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}p_K(t)\mathrm{d}t-\sum_{j=0}^{N-1}\int_0^T\frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}p_K(t)h(t)\mathrm{d}t\sum_{j=0}^{N-1}\int_0^T\frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}p_{K+1}(t)\mathrm{d}t,$$

we have $L_4(0)=0$ and

$$L_4'(T)=\sum_{j=0}^{N-1}\frac{(\theta T)^j}{j!}\mathrm{e}^{-\theta T}\sum_{j=0}^{N-1}\int_0^T\frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}\frac{[H(T)H(t)]^K}{K!(K+1)!}\times\mathrm{e}^{-H(T)-H(t)}[H(T)-H(t)][h(T)-(t)]\mathrm{d}t>0,$$

which follows that $H_2(N,K;T)$ increases strictly with K from $H_2(N,0;T)$ to $h(T)$ for any N and T. Therefore, because T is arbitrary, $H_2(N,K)\equiv H_2(N,K;\infty)$ increases strictly with N from $H_2(1,K)$ to $H_2(\infty,K) = H_1(K) = 1/\int_0^\infty p_K(t)\mathrm{d}t$ and increases strictly with K from $H_2(N,0)$ to $h(\infty)$, and because N is arbitrary, $H_1(K)$ increases strictly with K to $h(\infty)$.

Appendix C:

For $N = 0,1,2,\dots$, $K = 0,1,2,\dots$ and $0 < T < \infty$,

$$\widetilde{Q}_2(N,K;T) \equiv \frac{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t) h(t)\mathrm{d}t}{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t)\mathrm{d}t},$$

which increases strictly with N from $\widetilde{Q}_2(0,K;T)$ to $h(T)$ and increases strictly with K from $Q_2(N,\infty;T)$ to $h(T)$.
Proof. Note that

$$\widetilde{Q}_2(\infty,K;T) \equiv \lim_{N\to\infty} \frac{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t) h(t)\mathrm{d}t}{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t)\mathrm{d}t} = h(T),$$
$$\widetilde{Q}_2(N,\infty;T) \equiv \lim_{K\to\infty} \frac{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t) h(t)\mathrm{d}t}{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} P_K(t)\mathrm{d}t} = h(T).$$

Thus, by using the similar method of Appendix A, we can prove Appendix C. Therefore, $\widetilde{Q}_2(N,K) \equiv \widetilde{Q}_2(N,K;\infty)$ increases strictly with N from $\widetilde{Q}_2(0,K)$ to $h(\infty)$ and increases strictly with K form $Q_1(N)$ to $h(\infty)$.

Appendix D:

For $N = 0,1,2,\dots$, $K = 0,1,2,\dots$ and $0 < T < \infty$,

$$\widetilde{H}_2(N,K;T) = \frac{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t) h(t)\mathrm{d}t}{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t)\mathrm{d}t},$$

which increases strictly with N from $H_2(\infty,K;T)$ to $h(T)$ and increases strictly with K from $\widetilde{H}_2(N,0;T)$ to $h(T)$.
Proof. Note that

$$\widetilde{H}_2(\infty,K;T) = \lim_{N\to\infty} \frac{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t) h(t)\mathrm{d}t}{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t)\mathrm{d}t} = h(T),$$
$$\widetilde{H}_2(N,\infty;T) \equiv \lim_{K\to\infty} \frac{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t) h(t)\mathrm{d}t}{\sum_{j=N}^\infty \int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t} p_K(t)\mathrm{d}t} = h(T).$$

Thus, by using the similar method of Appendix B, we can prove Appendix D. Therefore, $\widetilde{H}_2(N,K) \equiv \widetilde{H}_2(N,K;\infty)$ increases strictly with N from $1/\int_0^\infty p_K(t)\mathrm{d}t$ to $h(\infty)$ and increases strictly with K from $\widetilde{H}_2(N,0)$ to $h(\infty)$.

Appendix E:

For $N = 0,1,2,\dots$ and $0 < T < \infty$,

$$Q_{O1}(N;T) \equiv \frac{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} h(t)\,\mathrm{d}t}{\int_0^T (\theta t)^N \mathrm{e}^{-\theta t} h(t)\left[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\right]\mathrm{d}t}$$

increases strictly with N from $Q_{O1}(0;T)$ to $Q_{O1}(\infty;T) = \overline{F}(T)/\int_T^\infty \overline{F}(t)\mathrm{d}t$.
Proof. Note that

$$Q_{O1}(\infty;T) \equiv \lim_{N\to\infty} Q_{O1}(N;T) = \frac{1}{\int_T^\infty \mathrm{e}^{-H(t)+H(T)}\mathrm{d}t} = \frac{\overline{F}(T)}{\int_T^\infty \overline{F}(t)\,\mathrm{d}t}.$$

Denoting

$$\begin{aligned} L_5(T) \equiv & \int_0^T (\theta t)^{N+1} \mathrm{e}^{-\theta t} h(t)\mathrm{d}t \\ & \times \int_0^T (\theta t)^N \mathrm{e}^{-\theta t} h(t)\Big[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\Big]\mathrm{d}t \\ & - \int_0^T (\theta t)^N \mathrm{e}^{-\theta t} h(t)\mathrm{d}t \\ & \times \int_0^T (\theta t)^{N+1} \mathrm{e}^{-\theta t} h(t)\Big[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u\Big]\mathrm{d}t, \end{aligned}$$

we have $L_5(0) = 0$ and

$$\begin{aligned} L_5'(T) = & (\theta T)^N \mathrm{e}^{-\theta T} h(T) \int_0^T (\theta t)^N \mathrm{e}^{-\theta t}(\theta T - \theta t) \\ & \times \Big[\int_t^\infty \mathrm{e}^{-H(u)+H(t)}\mathrm{d}u - \int_T^\infty \mathrm{e}^{-H(u)+H(T)}\mathrm{d}u\Big]\mathrm{d}t > 0, \end{aligned}$$

which follows that $Q_{O1}(N;T)$ increases strictly with N from $Q_{O1}(0;T)$ to $\overline{F}(T)/\int_T^\infty \overline{F}(t)\mathrm{d}t$ for any T. Thus, because T is arbitrary, $Q_{O1}(N) \equiv Q_{O1}(N;\infty)$ increases strictly with N from $Q_{O1}(0)$ to $\lim_{T\to\infty} \overline{F}(T)/\int_T^\infty \overline{F}(t)\mathrm{d}t = h(\infty)$.

Appendix F:

For $K = 0,1,2,\dots$ and $0 < T < \infty$,

$$H_{O1}(K;T) \equiv \frac{\int_0^T H(t)^K \mathrm{e}^{-H(t)}\Big[\int_0^\infty \theta \mathrm{e}^{-\theta u} h(t+u)\mathrm{d}u\Big]\mathrm{d}t}{\int_0^T H(t)^K \mathrm{e}^{-H(t)}\mathrm{d}t},$$

which increases strictly with K from $H_{O1}(0;T)$ to $\int_0^\infty \theta \mathrm{e}^{-\theta t}(T+t)\mathrm{d}t$.

Proof. Note that

$$H_{O1}(\infty;T) \equiv \lim_{K\to\infty} H_{O1}(K;T) = \int_0^\infty \theta \mathrm{e}^{-\theta t}(T+t)\,\mathrm{d}t,$$

Thus, by using the similar method of Appendix E, $H_{O1}(K;T)$ increases strictly with K from $H_{O1}(0;T)$ to $\int_0^\infty \theta \mathrm{e}^{-\theta t} h(T+t)\mathrm{d}t$ for any T. Therefore, because T is arbitrary, $H_{O1}(K) \equiv H_{O1}(K;\infty)$ increases strictly with K from $H_{O1}(0)$ to $\lim_{T\to\infty}\int_0^\infty \theta \mathrm{e}^{-\theta t}(T+t)\mathrm{d}t = h(\infty)$.

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