Which Replacement Is Better at Working Cycles or Number of Failures

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SUMMARY When a unit repeats some works over again and undergoes minimal repairs at failures, it is more practical to replace it preventively at the end of working cycles or at its failure times. In this case, it would be an interesting problem to know which is better to replace the unit at a number of working cycles or at random failures from the point of cost. For this purpose, we give models of the expected cost rates for the following replacement policies: (1) The unit is replaced at a working cycle $N$ and at a failure number $K$, respectively; (2) Replacement first and last policies with working cycle $N$ and failure number $K$, respectively; (3) Replacement overtime policies with working cycle $N$ and failure number $K$, respectively. Optimizations and comparisons of the policies for $N$ and $K$ are made analytically and numerically.

key words: replacement policies, minimal repair, working cycle, replacement first, replacement last, replacement overtime

1. Introduction

It has become an important problem to plan good maintenance policies for a large-scale system such as a plant equipment and an information system with network, as they have been widely used in various environments and their sudden failures might incur great losses or even social confusions. There have been many research works regarding to preventive maintenance policies in theory [1]–[4]; however, the difficulties are which maintenance policies are better from the points of cost, practicality and reliability. Further, if we could rank the maintenance policies as needed, but will this rank change when the maintenance environment changes. In order to answer the above questions, this paper tries to give some comparisons of replacement policies when an operating unit is replaced at working cycles or at random failures.

In general, it would be impossible to do some maintenance during the interval when the unit is operating for works, and it would be better to do maintenance when the work completes or when the unit fails. The theoretical models have shown that maintaining a unit after it completes some works are possible even though they are sometimes costly [5]. The properties of replacement times between two successive failed units were investigated for a system which replaced only at random times [6]. Several random and age replacement models were discussed [7] for an operating unit which repeats some works over again. The maintenance model with age $T$ and number of jobs completed $N$ has been considered [8]. Considering the systems successively executing jobs with random working times, it would be better to conduct the maintenance after the jobs completed. Other maintenance models with random working cycle have been studied extensively [9]–[11]. Furthermore, replacement first and last with two kinds of failures were considered and their optimal policies were discussed and compared [12]. Replacement policies in which a maintainer made the postponed replacement in a delay time due to inspection test were studied [13].

Nakagawa and Zhao considered about first and last policies for replacement and inspections when the policies are triggered by two factors. [14]–[16]. Replacement first means that the unit is replaced preventively at time of events such as operating time, number of repairs, working cycles, cumulative damage, etc, whichever occurs first, and replacement last means that the unit is replaced preventively at the above events, whichever occurs last. It has been shown that [15] replacement last policies could let the unit operate works as longer as possible while replacement first policies are more easily to save total maintenance cost. Replacement first and last policies are good alternatives when the unit performs a big project and the decision on replacement is based on the termination time of the project [10].

For replacement first and last policies, it is an interesting problem to determine which policies are better from the points of cost, practicality and reliability. The recent work [17] has given some comparative methods for replacement policies when they are performed at continuous or discrete times. However, when the unit can be only replaced preventively at discrete times such as working cycles or at random failures, this paper will answer the questions like how we can formulate the replacement first and last policies, and how we can know which policy is better from the point of cost. For this purpose, the following policies are given: (1) The unit is replaced at the $N$th $(N = 1, 2, \ldots)$ working cycle, and the $K$th $(K = 1, 2, \ldots)$ failure, respectively; (2) the unit is replaced at the $N$th working cycle or the $K$th failure, whichever occurs first and last. In addition, it is a possible way to delay the replacement policies over a planned time, whose models are called replacement overtime [18], so that the above policies are extended to overtime replacement policies with working cycles $N$ and failures $K$.  523
We formulate the expected cost rates, give analytical discussions and make comparisons to decided which is better for the above replacement policies in each sections. Finally, we give the properties of extended failure rates which are needed for theoretical analysis of optimal policies in Appendix.

2. Assumptions

We give the following assumptions for the models in this paper:

1. The unit repeats some works over again which have random working cycles $Y_j$ $(j = 1, 2, \ldots)$. It is assumed that $Y_j$ are independent and identically distributed random variables and have an identical distribution $G(t) \equiv \Pr\{Y \leq t\}$ with finite mean $1/\theta$ $\equiv \int_0^\infty G(t)dt$, where $G(t)$ is $1 - G(t)$. Let $G^{(n)}(t)$ $(n = 1, 2, \ldots)$ denote the $n$-fold Stieljes convolution of $G(t)$ and $G^{(0)}(t) \equiv 1$ for $t \geq 0$.

2. It is assumed that failures occur in a nonhomogeneous Poisson process with $H(t) \equiv \int_0^t h(u)du$. Let $p_j(t)$ denote the probability that the number of failures in $[0, t]$ is $j$, i.e.,

$$p_j(t) \equiv \frac{H(t)^j}{j!} e^{-H(t)} \quad (j = 0, 1, 2, \ldots).$$

We assume that the failure rate $h(t)$ increases strictly from $h(0) = 0$ to $h(\infty) = \infty$ for simplicity of discussions.

3. Let $P_{j+1}(t) \equiv \sum_{j=0}^\infty p_j(t)$ and $\overline{P}_{j+1}(t) \equiv \sum_{j=0}^\infty p_j(t)$ $(K = 0, 1, 2, \ldots)$, where note that $\sum_{j=0}^\infty p_j(t) = 1$. Further, we have the following relations for $t (0 < t < \infty)$ and $j (j = 0, 1, 2, \ldots)$,

$$P_{j+1}(t) = \int_0^t p_j(u)h(u)du,$$

$$\overline{P}_{j+1}(t) = \int_0^\infty p_j(u)h(u)du, \quad \int_0^\infty p_j(t)h(t)dt = 1,$$

$$\int_0^H(t) dP_j(t) = \int_0^\infty \overline{P}_j(t)h(t)dt = \sum_{j=0}^{j-1} \int_0^\infty p_j(t)h(t)dt = j.$$

4. The probability that some failures occur in $(0, t]$ is given by $F(t) \equiv \sum_{j=1}^\infty p_j(t) = 1 - p_0(t) = 1 - e^{-H(t)}$ with finite mean $\mu$, and $f(t)$ is a density function of $F(t)$ and $f(t) \equiv \frac{dF(t)}{dt}$. Thus, for given $t (0 \leq t < \infty)$, the probability that a failure occurs in $(u, u+du]$ is $f(u)du/F(t)$ for $u > t$.

5. When the failure has occurred, the unit is replaced or undergoes minimal repair. The unit after minimal repair has the same failure rate as before rate [3, p. 96].

3. Basic Policies

3.1 Replacement at Cycle $N$

Suppose that the unit is replaced at working cycle $N$ ($N = 1, 2, \ldots$) (see Fig. 1). Then, the expected cost rate is [5, p. 76].

$$C(N) = \frac{c_N + c_M \int_0^N \frac{1 - G^{(N)}(t)}{N/\theta} h(t)dt}{N/\theta} \quad (N = 1, 2, \ldots),$$

where $c_N = \text{replacement cost at cycle } N$ and $c_M = \text{cost of minimal repair at each failure}$.

We find optimal $N^*$ to minimize $C(N)$. Forming the inequality $C(N + 1) - C(N) \geq 0$,

$$\int_0^N \frac{1 - G^{(N)}(t)}{N/\theta} h(t)dt \geq \frac{c_N}{c_M},$$

where for $0 < T \leq \infty$ and $N = 0, 1, 2,\ldots$,

$$Q_1(N; T) = \int_0^T \left( \frac{G^{(N)}(t) - G^{(N+1)}(t)}{h(t)} \right) dt \leq h(T),$$

$$Q_1(N) = \lim_{T \to \infty} Q_1(N; T) = \theta \int_0^N \left( \frac{G^{(N)}(t) - G^{(N+1)}(t)}{h(t)} \right) dt.$$  

In particular, when $G(t) = 1 - e^{-\theta t}$,

$$Q_1(N) = \frac{\theta^N}{N!} e^{-\theta t} h(t)dt,$$

which increases strictly with $N$ to $\infty$ from Appendix A. Thus, there exists a finite and unique minimum $N^*$ ($1 \leq N^* < \infty$) which satisfies (3), and the resulting cost rate is $c_MQ_1(N^* - 1) < C(N^*) \leq c_MQ_1(N^*)$.

3.2 Replacement at Failure $K$

Suppose that the unit is replaced at failure $K$ ($K = 1, 2, \ldots$) (see Fig. 2). Then, the expected cost rate is [3, p. 106]

$$C(K) = \frac{c_K + c_M K}{\int_0^\infty \overline{P}_K(t)dt} \quad (K = 1, 2, \ldots),$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{replacement.png}
\caption{Replacement at cycle $N$.}
\end{figure}
where \( c_K = \) replacement cost at failure \( K \).

We find optimal \( K^* \) to minimize \( C(K) \). Forming the inequality \( C(K + 1) - C(K) \geq 0 \),

\[
H_1(K) \int_0^\infty \bar{P}_k(t)dt - K \geq \frac{c_K}{c_M},
\]

(6)

where for \( 0 < T \leq \infty \) and \( K = 0, 1, 2, \ldots, \)

\[
H_1(K; T) = \int_0^T p_K(t)h(t)dt
\]

and

\[
H_1(K) = \lim_{T \to \infty} H_1(K; T) = \frac{1}{\int_0^\infty p_K(t)dt},
\]

which increases strictly with \( K \) to \( h(\infty) \) from Appendix B. Thus, because the left-hand side of (6) increases strictly with \( K \) to \( \infty \), there exists a finite and unique minimum \( K^* \) (1 \( \leq \) \( K^* \) \( \leq \) \( K \)) which satisfies (6), and the resulting cost rate is

\[
c_mH_1(K^* - 1) < C(K^*) \leq c_mH_1(K^*). \]

(7)

3.3 Numerical Examples

We compute numerically optimal \( N^* \) and \( K^* \) when \( G(t) = 1 - e^{-t} \) and \( H(t) = (\lambda t)^2 \), i.e., \( h(t) = 2\lambda t^2 \). In this case,

\[
Q_1(N) = \int_0^{\infty} \int_0^{\infty} \frac{t^N}{N!} e^{-2\lambda t^2} dt \equiv 2\lambda^2(N + 1),
\]

\[
\sum_{j=0}^{N-1} \int_0^{\infty} \frac{t^j}{j!} e^{-2\lambda t^2} dt = \lambda^2N(N + 1),
\]

and from (3), optimal \( N^* \) is given by

\[
2\lambda^2N(N + 1) = \lambda^2N(N + 1) \geq \frac{c_N}{c_m}.
\]

(8)

and from (2),

\[
\frac{C(N^*)}{c_m} = \frac{c_N/c_m + \lambda^2N(N^* + 1)}{N^*}.
\]

(9)

Further, we can see that

\[
\int_0^{\infty} p_K(t)dt = \int_0^{\infty} \frac{(\lambda t)^2}{K!} e^{-\lambda t^2} dt = \frac{1}{\Gamma(K + 1/2)} \frac{\Gamma(K + 1/2)}{\Gamma(K + 1)}
\]

\[
\sum_{j=0}^{K-1} \int_0^{\infty} p_j(t)dt = \frac{1}{\lambda} \frac{\Gamma(K + 1/2)}{\Gamma(K)}
\]

and from (6), optimal \( K^* \) is given by

\[
\frac{2\Gamma(K + 1/2)/\Gamma(K)}{\Gamma(K + 1/2)/\Gamma(K + 1)} - K = K^* \geq \frac{c_K}{c_m}.
\]

(10)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
& \( c_m \) & \( K^* \) & \( N^* \) & \( C(N^*)/c_m \) & \( C(N^*)/\tilde{c}_m \) & \( \tilde{C}_N \) & \( \tilde{C}_N^* \) \\
\hline
1 & 1 & 10 & 0.226 & 0.210 & 1 & 2.257 & 3.000 \\
2 & 2 & 14 & 0.301 & 0.293 & 1 & 3.009 & 4.000 \\
3 & 3 & 17 & 0.361 & 0.357 & 2 & 3.611 & 4.500 \\
4 & 4 & 20 & 0.413 & 0.410 & 2 & 4.127 & 5.000 \\
5 & 5 & 22 & 0.459 & 0.457 & 2 & 4.858 & 5.500 \\
6 & 6 & 26 & 0.500 & 0.503 & 2 & 5.002 & 6.000 \\
7 & 7 & 24 & 0.559 & 0.542 & 3 & 5.387 & 6.333 \\
8 & 8 & 26 & 0.575 & 0.578 & 3 & 5.746 & 6.667 \\
9 & 9 & 28 & 0.608 & 0.611 & 3 & 6.084 & 7.000 \\
10 & 10 & 32 & 0.640 & 0.643 & 3 & 6.404 & 7.333 \\
\hline
\end{tabular}
\caption{Optimal \( N^* \), \( K^* \), \( C(K^*)/c_m \) and \( C(N^*)/c_m \) when \( G(t) = 1 - e^{-t} \), \( H(t) = (\lambda t)^2 \) and \( \tilde{c}_N = \tilde{c}_K \).}
\end{table}

where \( \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \) for \( \alpha > 0 \). Thus, if \( c_K/c_m \) is an integer then \( K^* = c_K/c_m \), and from (5),

\[
C(K^*) = \frac{c_K}{c_m} + K^*
\]

\[
\Gamma(K^* + 1/2) / \Gamma(K^* + 1/2)
\]

(11)

Table 1 gives optimal \( K^* \), \( N^* \), \( C(K^*)/c_m \) and \( C(N^*)/c_m \) for \( \lambda = 0.1, 1 \), and \( \tilde{c}_N/c_m = 1, 2, \ldots, 10 \). We can see that for \( \lambda = 1 \), \( C(K^*)/c_m < C(N^*)/c_m \), that is, replacement with \( K^* \) is better than replacement with \( N^* \). On the other hand, for \( \lambda = 0.1, C(K^*)/c_m > C(N^*)/c_m \) for \( K^* = c_N/c_m \leq 5 \), and \( C(K^*)/c_m < C(N^*)/c_m \) for \( K^* \geq 6 \). Optimal \( N^* \) decreases with \( \lambda \). The reason would be that when \( \lambda \) is large, interval times of failures become small and we should replace early to avoid the cost of failures.

4. Replacement First and Last

4.1 Replacement First

The unit is replaced at cycle \( N (N = 1, 2, \ldots) \) or failure \( K (K = 1, 2, \ldots) \), whichever occurs first (see Fig.3). The probability that the unit is replaced at cycle \( N \) is \( \int_0^\infty \bar{P}_k(t)G(N)(t) \), and the probability that it is replaced at failure \( K \) is \( \int_0^\infty \frac{G(N)}{N} \) \( \bar{P}_k(t) \). The mean time to replacement is

\[
\int_0^\infty t \bar{P}_k(t)G(N)(t) + \int_0^\infty t \bar{P}_k(t)G(N)(t) \bar{P}_k(t)
\]

\[
\int_0^\infty [1 - G(N)(t)] \bar{P}_k(t)dt,
\]

and the expected number of failures until replacement is

\[
\int_0^\infty H(t)\bar{P}_k(t)G(N)(t) + \int_0^\infty H(t)[1 - G(N)(t)] \bar{P}_k(t)
\]

\[
\int_0^\infty [1 - G(N)(t)] \bar{P}_k(t)dt.
\]

Therefore, the expected cost rate is

\[
C_f(N, K) = \frac{c_K - c_N}{c_K - c_N} \int_0^\infty \bar{P}_k(t)G(N)(t)
\]

\[
\int_0^\infty [1 - G(N)(t)] \bar{P}_k(t)dt.
\]

(14)
Clearly, \( C_F(\infty, K) = C(K) \) in (5) and \( C_F(N, \infty) = C(N) \) in (2).

We find optimal \( N_F^* \) and \( K_F^* \) to minimize \( C_F(N, K) \) when \( c_K = c_N \) and \( G(t) = 1 - e^{-\theta t} \). Forming the inequality \( C_F(N + 1, K) - C_F(N, K) \geq 0 \),

\[
Q_2(N, K) \int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt
- \int_0^\infty [1 - G^{(N)}(t)]P_K(t)h(t)dt \geq \frac{c_N}{c_M},
\]

where

\[
Q_2(N, K) \equiv \sum_{j=0}^{K-1} \int_0^\infty (\theta t)^N e^{-\theta t} p_j(t)h(t)dt \sum_{j=0}^{K-1} \int_0^\infty (\theta t)^N e^{-\theta t} p_j(t)dt,
\]

which increases strictly with \( N \) from \( Q_2(0, K) \) to \( h(\infty) \) and increases strictly with \( K \) from \( Q_2(N, 1) \) to

\[
Q_2(N, \infty) = \int_0^\infty \theta t \theta t^N e^{-\theta t}h(t)dt = Q_1(N)
\]

from Appendix A. Thus, because the left-hand side of (13) increases strictly with \( N \) to \( h(\infty) \), there exists a finite and unique minimum \( N_F^* \) which satisfies (15), and the resulting cost rate is

\[
c_M Q_2(N_F^* - 1, K) < C_F(N_F^*, K) \leq c_M Q_2(N_F^*, K).
\]

In addition, noting that the left-hand side of (15) goes to that of (3) as \( K \to \infty \), \( N_F^* \) approaches to \( N^* \) given in (3) as \( K \to \infty \).

Forming the inequality \( C_F(N + 1, K) - C_F(N, K) \geq 0 \),

\[
H_2(N, K) \int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt
- \int_0^\infty [1 - G^{(N)}(t)]P_K(t)h(t)dt \geq \frac{c_N}{c_M},
\]

where

\[
H_2(N, K) \equiv \sum_{j=0}^{N-1} \int_0^\infty [(\theta t)^j/j!]e^{-\theta t} p_N(t)h(t)dt \sum_{j=0}^{N-1} \int_0^\infty [(\theta t)^j/j!]e^{-\theta t} p_N(t)dt,
\]

which increases strictly with \( N \) from \( H_2(K, 1) \) to

\[
H_2(K, \infty) = \int_0^\infty p_N(t)h(t)dt \int_0^\infty p_N(t)dt = H_1(K),
\]

and increases strictly with \( K \) from \( H_2(0, N) \) to \( h(\infty) \) from

Appendix B. Thus, because the left-hand side of (17) increases strictly with \( K \) to \( \infty \), there exists a finite and unique minimum \( K_F^* \) which satisfies (17), and the resulting cost rate is

\[
c_M H_2(K_F^* - 1, N) < C_F(N, K_F^*) \leq c_M H_2(K_F^*, N).
\]

In addition, noting that the left-hand side of (17) goes to that of (6) as \( N \to \infty \), \( K_F^* \) approaches to \( K^* \) given in (6) as \( N \to \infty \).

4.2 Replacement Last

The unit is replaced at cycle \( N \) (\( N = 0, 1, 2, \ldots \)) or failure \( K \) (\( K = 0, 1, 2, \ldots \)), whichever occurs last (see Fig. 4). The probability that the unit is replaced at cycle \( N \) is \( \int_0^\infty P_K(t)dg^{(N)}(t) \) and the probability that it is replaced at failure \( K \) is \( \int_0^\infty G^{(N)}(t)dp_K(t) \). The mean time to replacement is

\[
\int_0^\infty t P_K(t)dg^{(N)}(t) + \int_0^\infty t G^{(N)}(t)dp_K(t)
= \int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt,
\]

and the expected number of failures until replacement is

\[
\int_0^\infty H(t)P_K(t)dg^{(N)}(t) + \int_0^\infty H(t)G^{(N)}(t)dp_K(t)
= \int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt.
\]

Therefore, the expected cost rate is

\[
C_L(N, K) = \frac{c_K - (c_N - c_K) \int_0^\infty P_K(t)dg^{(N)}(t)}{\int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt}.
\]

Clearly, \( C_L(0, K) = C(K) \) in (5) and \( C_L(N, 0) = C(N) \) in (2).

We find optimal \( N_L^* \) and \( K_L^* \) to minimize \( C_L(N, K) \) when \( c_K = c_N \) and \( G(t) = 1 - e^{-\theta t} \). Forming the inequality \( C_L(N + 1, K) - C_L(N, K) \geq 0 \),

\[
\widetilde{Q}_2(N, K) \int_0^\infty [1 - G^{(N)}(t)]P_K(t)dt
- \int_0^\infty [1 - G^{(N)}(t)]P_K(t)h(t)dt \geq \frac{c_K}{c_M},
\]

where
\[
\bar{Q}_2(N, K) = \sum_{j=K}^{\infty} \frac{(\theta t)^N e^{-\theta t^e} p_j(t)h(t)dt}{N!},
\]
which increases strictly with \(N\) from \(\bar{Q}_2(0, K)\) to \(h(\infty)\) and increases strictly with \(K\) from
\[
\bar{Q}_2(N, 0) = \int_0^\infty \frac{\theta t^N e^{-\theta t^e} h(t)dt}{N!},
\]
to \(h(\infty)\), and \(\bar{Q}_2(N, K)\) from Appendix C. Thus, because the left-hand side of (22) increases strictly with \(N\) to \(\infty\), there exists a finite and unique minimum \(N^*_L(0 \leq N^*_L < \infty)\) which satisfies (22), and the resulting cost rate is
\[
c_M \bar{Q}_2(N^*_L - 1, K - 1) < C_L(N^*_L, K) \leq c_M \bar{Q}_2(N^*_L, K - 1).
\]
(23)

In addition, noting that the left-hand side of (22) agrees with that of (3) when \(K = 0, N^*_L = N^*\) given in (3) when \(K = 0\).

Forming the inequality \(C_L(N, K + 1) - C_L(N, K) \geq 0,\)
\[
\bar{H}_2(K, N) \sum_{t=K}^{\infty} [1 - G(N)(t)]p_K(t)dt
- \sum_{t=0}^{\infty} [1 - G(N)(t)]h(t)dt \geq c_K^{-} / c_M^{-}
\]
where
\[
\bar{H}_2(K, N) = \sum_{t=K}^{\infty} G(N)(t)p_K(t)h(t)dt \sum_{t=K}^{\infty} G(N)(t)p_K(t)dt
\]
which increases strictly with \(K\) from \(\bar{H}_2(0, N)\) to \(h(\infty)\) from Appendix D. Thus, because the left-hand side of (24) increases strictly with \(K\) to \(\infty\), there exists a finite and unique minimum \(K^*_L(0 \leq K^*_L < \infty)\) which satisfies (24), and the resulting cost rate is
\[
c_M \bar{H}_2(K^*_L - 1, N) < C_L(N, K^*_L) \leq c_M \bar{H}_2(K^*_L, N).
\]
(25)

In addition, noting that the left-hand side of (24) agrees with that of (6) when \(N = 0, K^*_L = K^*\) given in (6) when \(N = 0\).

We compute numerically optimal \((K^*_L, N^*_L)\) and \((K^*_L, N^*_L)\) when \(c_M^{-} = c_K^{-}\) and \(H(t) = (\lambda t)^2\). Tables 2 and 3 give \((K^*_L, N^*_L), C_F(K^*_L, N^*_L) / c_M, (K^*_L, N^*_L)\) and \(C_L(K^*_L, N^*_L) / c_M\) for \(c_K / c_M = 1, 2, \ldots, 10\) when \(\lambda = 0.1\) and \(\lambda = 1.0\), respectively. We can see from these tables that \(C_F(K^*_L, N^*_L) / c_M < C_L(K^*_L, N^*_L) / c_M\).

5. Overtime Policies

5.1 Replacement at First Failure Over Cycle \(N\)

Suppose that the unit is replaced at the first failure over working cycle \(N\) \((N = 0, 1, 2, \ldots, N)\) (see Fig. 5). Recall that \(F(t) = 1 - e^{-H(t)}\) and the probability that the unit with age \(t\) fails in \((u, u + du]\) for \(u > t\) is \(f(u)du / F(t)\). Thus, the mean time to replacement is
\[
\int_0^\infty \frac{1}{F(t)} \left[ \int_t^\infty u dF(u) \right] dG_N(t)
= \frac{N}{\theta} + \int_0^\infty \left[ \int_t^\infty e^{-H(u) + H(t)} du \right] dG_N(t)
= \mu + \int_0^\infty \left[ 1 - G_N(t) \right] \left[ \int_t^\infty e^{-H(u) + H(t)} du \right] h(t)dt,
\]
and the expected number of failures until replacement is
\[
\int_0^\infty \frac{1}{F(t)} \left[ \int_t^\infty H(u) dF(u) \right] dG_N(t)
= 1 + \int_0^\infty \left[ 1 - G_N(t) \right] h(t)dt.
\]
Thus, the expected cost rate is
\[
C_O(N) = \frac{c_{ON} + c_M \left[ 1 + \int_0^\infty \left[ 1 - G_N(t) \right] h(t)dt \right]}{\mu + \int_0^\infty \left[ 1 - G_N(t) \right] \left[ \int_t^\infty e^{-H(u) + H(t)} du \right] h(t)dt},
\]
where \(c_{ON}\) = replacement cost at first failure over cycle \(N\).

We find optimal \(N^*_O\) to minimize \(C_O(N)\) when \(G(t) = 1 - e^{-\lambda t}\).
1 - e^{-\theta t}. Forming the inequality \( C_0(N + 1) - C_0(N) \geq 0, \)
\[
Q_{01}(N) \left\{ \mu + \int_0^\infty \left[ 1 - G^{(N)}(t) \right] \int_0^\infty e^{-H(u) + H(t)} \, du \right\} dr
\]
\[- \int_0^\infty \left[ 1 - G^{(N)}(t) \right] h(t) \, dt - 1 \geq \frac{c_{\text{OK}}}{c_M}, \tag{29}
\]
where
\[
Q_{01}(N) \equiv \int_0^\infty (\theta N e^{-\theta h(t)}) \, dt
\]
which increases strictly with \( N \) to \( h(\infty) \) from Appendix E. Thus, because the left-hand side of (29) increases strictly with \( N \) to \( \infty \), there exists a finite and unique minimum \( N_0^* \) (\( 0 \leq N_0^* < \infty \)) which satisfies (29), and the resulting cost rate is
\[
c_M Q_{01}(N_0^* - 1) < C_0(N_0^*) \leq c_M Q_{01}(N_0^*). \tag{30}
\]

5.2 Replacement at First Cycle Over Failure \( K \)

Suppose that the unit is replaced at the first working cycle over failure \( K \) (\( K = 0, 1, 2, \ldots \) ) (see Fig. 6). The mean time to replacement is
\[
\sum_{j=0}^\infty \int_0^\infty \left\{ \int_0^\infty \left[ \int_0^{\infty} y \, dG(y - u) \right] e^{G^{(j)}(u)} \right\} \, dP_K(t)
\]
\[
= \int_0^\infty P_K(t) \, dt
\]
\[
+ \sum_{j=0}^\infty \int_0^\infty \left\{ \int_0^\infty G(y - u) \, dy \right\} e^{G^{(j)}(u)} \, dP_K(t)
\]
\[
= \frac{1}{\theta} \sum_{j=0}^\infty \int_0^\infty G^{(j)}(t) \, dP_K(t), \tag{31}
\]
and the expected number of failures until replacement is
\[
\sum_{j=0}^\infty \int_0^\infty \left\{ \int_0^\infty H(y) \, dG(y - u) \right\} e^{G^{(j)}(u)} \, dP_K(t)
\]
\[
= \sum_{j=0}^\infty \int_0^\infty \left\{ \int_0^\infty G(y) \, h(u + y) \, dy \right\} e^{G^{(j)}(u)} \, dP_K(t). \tag{32}
\]

Therefore, the expected cost rate is
\[
C_{\text{OK}}(K) =
\frac{c_{\text{OK}} + c_M \sum_{j=0}^\infty \left\{ \int_0^\infty \left[ \int_0^\infty G(y) \, h(u + y) \, dy \right] e^{G^{(j)}(u)} \right\} \, dP_K(t)}{(1/\theta) \sum_{j=0}^\infty \int_0^\infty G^{(j)}(t) \, dP_K(t)}, \tag{33}
\]
where \( c_{\text{OK}} \) = replacement cost at first cycle over failure \( K \). In particular, when \( G(t) = 1 - e^{-\theta t} \),
\[
C_{\text{OK}}(K) =
\frac{c_{\text{OK}} + c_M \left[ \int_0^\infty e^{-\theta h(t)} \, dt + \int_0^\infty P_K(t) \int_0^\infty \theta e^{-\theta h(t + u)} \, du \, dt \right]}{1/\theta + \int_0^\infty P_K(t) \, dt}. \tag{34}
\]

We find optimal \( K^*_0 \) to minimize \( C_{\text{OK}}(K) \) in (34). Forming the inequality \( C_{\text{OK}}(K + 1) - C_{\text{OK}}(K) \geq 0, \)
\[
H_{\text{OK}}(K) \left\{ \frac{1}{\theta} \right\} + \int_0^\infty P_K(t) \, dt - \int_0^\infty e^{-\theta h(t)} \, dt
\]
\[
- \int_0^\infty \int_0^\infty P_K(t) \left[ \int_0^\infty \theta e^{-\theta h(t + u)} \, du \right] \, dt \geq \frac{c_{\text{OK}}}{c_M}, \tag{35}
\]
where for \( 0 < T \leq \infty, \)
\[
H_{\text{OK}}(K, T) \equiv \int_0^T H(t) e^{-\theta h(t)} \left[ \int_0^\infty \theta e^{-\theta h(t + u)} \, du \right] \, dt
\]
\[
H_{\text{OK}}(K) \equiv \lim_{T \to \infty} H_{\text{OK}}(K, T)
\]
\[
= \int_0^\infty H(t) e^{-\theta h(t)} \left[ \int_0^\infty \theta e^{-\theta h(t + u)} \, du \right] \, dt
\]
\[
\int_0^\infty H(t) e^{-\theta h(t)} \, dt,
\]
which increases strictly with \( K \) to \( \int_0^\infty \theta e^{-\theta h(t + u)} \, du \, dt \) from Appendix F. Thus, because the left-hand side of (35) increases strictly with \( K \) to \( \infty \), there exists a finite and unique minimum \( K_0^* \) (\( 0 \leq K_0^* < \infty \)) which satisfies (35), and the resulting cost rate is
\[
c_M H_{\text{OK}}(K_0^* - 1) < C_{\text{OK}}(K_0^*) \leq c_M H_{\text{OK}}(K_0^*). \tag{36}
\]

We discuss numerically optimal \( N_0^*, K_0^* \) when \( c_{\text{OK}} = c_{\text{OK}}, G(t) = 1 - e^{-t} \) and \( H(t) = (\lambda t)^2 \). Tables 4 and 5 give optimal \( N_0^*, C_{\text{OK}}(N_0^*), K_0^*, C_{\text{OK}}(K_0^*) \) when \( \lambda = 0.1 \) and \( \lambda = 1 \), respectively. We can see from these tables that \( C_{\text{OK}}(N_0^*) < C_{\text{OK}}(K_0^*) \) for \( c_{\text{OK}}/c_M \leq 5 \) and \( C_{\text{OK}}(N_0^*) \geq C_{\text{OK}}(K_0^*) \) for \( c_{\text{OK}}/c_M \geq 6 \) in Table 4, \( C_{\text{OK}}(N_0^*) \leq C_{\text{OK}}(K_0^*) \) for \( c_{\text{OK}}/c_M \leq 3 \) and \( C_{\text{OK}}(N_0^*) \geq C_{\text{OK}}(K_0^*) \) for \( c_{\text{OK}}/c_M \leq 5 \) in Table 5. This means that \( C_{\text{OK}}/c_M \) has a threshold level and if \( c_{\text{OK}}/c_M \) is smaller than this level, i.e., replacement cost is relatively

---

**Table 4** Optimal \( N_0^*, C_{\text{OK}}(N_0^*)/c_M, K_0^*, C_{\text{OK}}(K_0^*)/c_M \), when \( G(t) = 1 - e^{-t}, H(t) = (\lambda t)^2 \) and \( c_{\text{OK}} = c_{\text{OK}}, \lambda = 0.1 \).

<table>
<thead>
<tr>
<th>( c_{\text{OK}}/c_M )</th>
<th>( N_0^* )</th>
<th>( C_{\text{OK}}(N_0^*)/c_M )</th>
<th>( K_0^* )</th>
<th>( C_{\text{OK}}(K_0^*)/c_M )</th>
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<td>0.575</td>
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<td>0.643</td>
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<td>0.641</td>
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smaller against minimal repair cost $c_M$, replacement at first failure over cycle $N$ is better than replacement at first cycle over failure $K$.

6. Conclusions

We have discussed theoretically and numerically the optimal policies of replacements with cycle $N$ and failure $K$. In general, it would be more difficult to derive theoretically optimal policies for replacements with discrete variables than those with continuous ones. This paper has given several mathematical techniques of solving optimization problems with discrete variables and these would be useful for maintainances of actual models in practical fields. For example, we can propose the following replacement policies from the results of this paper:

1. If $c_N$ is smaller than $c_K$ and cycle $N$ can be counted more easily than failure $K$, then cycle $N$ is better than failure $K$.

2. If $c_K$ is smaller than $c_N$ and failure $K$ can be counted more easily than cycle $N$, then failure $K$ is better than cycle $N$.

3. If $c_K$ is smaller than $c_N$ and cycle $N$ can be counted more easily than failure $K$, then replacement overtime with cycle $N$ is better than failure $K$.

4. If $c_N$ is smaller than $c_K$ and failure $K$ can be counted more easily than cycle $N$, then replacement overtime with failure $K$ is better than cycle $N$.

5. If both costs of $c_N$ and $c_K$ are almost the same and cycle $N$ and failure $K$ can be counted easily, we compute the expected costs $C(N^*)$ and $C(K^*)$, and the expected costs $C(N^*, K^*_N)$ and $C(N^*_K, K^*_F)$ numerically, and decide the optimal replacement policy.

The replacement policies proposed in this paper would be applied to the cumulative damage models and data backup models of computer systems by making some suitable modifications [19], [20].

Acknowledgments

This work is supported by JSPS KAKENHI Grant Number 18K01713, National Natural Science Foundation of China (NO. 71801126), Natural Science Foundation of Jiangsu Province (NO. BK20180412) and Fundamental Research Funds for the Central Universities (NO. NR2018003).

References


Appendix A:

For $N = 0, 1, 2, \ldots$, $K = 1, 2, \ldots$ and $0 < T < \infty$, 

$$Q_2(N, K; T) = \frac{\int_0^T \left(\theta t\right)^N e^{-\lambda t} \mathcal{P}_K(t) h(t) \, dt}{\int_0^T \left(\theta t\right)^N e^{-\lambda t} \mathcal{P}_K(t) \, dt},$$

Table 5: Optimal $N_0^*$, $C_0(N_0^*)/c_M$, $K_0^*$, $C_0(K_0^*)/c_M$, when $G(t) = 1 - e^{-\lambda t}$, $H(t) = (\theta t)^N$ and $c_{ON} = c_{OK}$. $\lambda = 1$.

<table>
<thead>
<tr>
<th>$c_{ON}/c_M$</th>
<th>$N_0^*$</th>
<th>$C_0(N_0^*)/c_M$</th>
<th>$K_0^*$</th>
<th>$C_0(K_0^*)/c_M$</th>
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</table>
which increases strictly with \(N\) from \(Q_2(0, K; T)\) to \(h(T)\) and increases strictly with \(K\) from \(Q_2(N, 1; T)\) to
\[
Q_2(N; \infty; T) = \frac{\int_0^T (\theta)^N e^{-\theta h(t)} \, dt}{\int_0^T (\theta)^N e^{-\theta h(t)} \, dt}.
\]

**Proof.** Note that
\[
Q_2(\infty; K; T) = \lim_{N \to \infty} \frac{\int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)h(t)} \, dt}{\int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt} = h(T),
\]
\[
Q_2(N; \infty; T) = \frac{\int_0^T (\theta)^N e^{-\theta h(t)} \, dt}{\int_0^T (\theta)^N e^{-\theta h(t)} \, dt}.
\]

Denoting
\[
L_1(T) \equiv \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt - \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt,
\]
we have \(L_1(0) = 0\) and
\[
L_1'(T) = (\theta \mathcal{P}_K(t)) \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt = (\theta \mathcal{P}_K(t)) \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_K(t)} \, dt > 0,
\]
which follows that \(Q_2(N, K; T)\) increases strictly with \(N\) from \(Q_2(0, K; T)\) to \(h(T)\) for any \(K\) and \(T\). Similarly, denoting
\[
L_2(T) \equiv \sum_{j=0}^{K-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt \sum_{j=0}^{K-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt - \sum_{j=0}^{K-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt \sum_{j=0}^{K-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt,
\]
we have \(L_2(0) = 0\) and
\[
L_2'(T) = (\theta \mathcal{P}_K(t)) \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt - \sum_{j=0}^{K-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt > 0,
\]
which follows that \(Q_2(N, K; T)\) increases strictly with \(K\) from \(Q_2(0, K; T)\) to \(h(T)\) and increases strictly with \(K\) from \(Q_2(N, 1; T)\) to \(Q_1(N) = \theta \int_0^T (\theta)^N e^{-\theta h(t)} \, dt\), and because \(K\) is arbitrary, \(Q_1(N)\) increases strictly with \(N\) to \(h(\infty)\).

**Appendix B:**

For \(N = 1, 2, \ldots, K = 0, 1, 2, \ldots\) and \(0 < T < \infty\),
\[
H_2(N, K; T) = \frac{\sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt}{\sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt} p_K(t)h(t) \, dt,
\]
which increases strictly with \(N\) from \(H_2(1, K; T)\) to
\[
H_2(\infty, K; T) = \int_0^T p_K(t)h(t) \, dt,
\]
and increases strictly with \(N\) from \(H_2(0, N; T)\) to \(h(T)\). **Proof.** Note that
\[
H_2(\infty, K; T) = \int_0^T p_K(t)h(t) \, dt,
\]
\[
H_2(N, \infty; T) \equiv \lim_{K \to \infty} \frac{\sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt}{\int_0^T (\theta)^j \, dt} p_K(t)h(t) \, dt = h(T),
\]
Denoting
\[
L_3(T) \equiv \int_0^T (\theta)^N (\theta^j \, dt) p_K(t)h(t) \, dt \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt - \int_0^T (\theta)^N (\theta^j \, dt) p_K(t)h(t) \, dt \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt,
\]
we have \(L_3(0) = 0\) and
\[
L_3'(T) = \sum_{j=0}^{N-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt - \sum_{j=0}^{N-1} \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt \int_0^T (\theta)^N e^{-\theta \mathcal{P}_j(t)} \, dt > 0,
\]
which follows that \(H_2(N, K; T)\) increases strictly with \(N\) from \(H_2(1, K; T)\) to \(H_2(\infty, K; T)\) for any \(K\) and \(T\). Similarly, denoting
\[
L_4(T) \equiv \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt p_K(t) \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt - \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt p_K(t) \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt,
\]
we have \(L_4(0) = 0\) and
\[
L_4'(T) = \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt e^{-\theta \mathcal{P}_j(t)} \int_0^T (\theta)^j \, dt e^{-\theta \mathcal{P}_j(t)} - \sum_{j=0}^{N-1} \int_0^T (\theta)^j \, dt e^{-\theta \mathcal{P}_j(t)} \int_0^T (\theta)^j \, dt e^{-\theta \mathcal{P}_j(t)} > 0,
\]
which follows that \(H_2(N, K; T)\) increases strictly with \(K\) from \(H_2(N, 0; T)\) to \(h(T)\) for any \(N\) and \(T\). Therefore, because \(T\) is arbitrary, \(H_2(N, K) \equiv H_2(N, K; \infty)\) increases
strictly with $N$ from $H_2(1, K)$ to $H_2(\infty, K) = H_1(K) = 1/\int_0^\infty p_K(t)dt$ and increases strictly with $K$ from $H_2(N, 0)$ to $h(\infty)$, and because $N$ is arbitrary, $H_1(K)$ increases strictly with $K$ to $h(\infty)$.

Appendix C:

For $N = 0, 1, 2, \ldots, K = 0, 1, 2, \ldots$ and $0 < T < \infty$,

$$\tilde{Q}_2(N, K; T) = \lim_{N \to \infty} \frac{\int_0^T (\theta t)^N e^{-\theta t} P_K(t)h(t)dt}{\int_0^\infty (\theta t)^Ne^{-\theta t}p_K(t)dt},$$

which increases strictly with $N$ from $\tilde{Q}_2(0, K; T)$ to $h(T)$ and increases strictly with $K$ from $\tilde{Q}_2(N, \infty; T)$ to $h(T)$.

Proof. Note that

$$\tilde{Q}_2(\infty, K; T) = \lim_{N \to \infty} \frac{\int_0^T (\theta t)^N e^{-\theta t} P_K(t)h(t)dt}{\int_0^\infty (\theta t)^Ne^{-\theta t}p_K(t)dt} = h(T),$$

$$\tilde{Q}_2(N, \infty; T) = \lim_{K \to \infty} \frac{\int_0^T (\theta t)^N e^{-\theta t} P_K(t)h(t)dt}{\int_0^\infty (\theta t)^Ne^{-\theta t}p_K(t)dt} = h(T).$$

Thus, by using the similar method of Appendix A, we can prove Appendix C. Therefore, $Q_2(N, K) \equiv \tilde{Q}_2(N, K; \infty)$ increases strictly with $N$ from $Q_2(0, K)$ to $h(\infty)$ and increases strictly with $K$ from $Q_1(N)$ to $h(\infty)$.

Appendix D:

For $N = 0, 1, 2, \ldots, K = 0, 1, 2, \ldots$ and $0 < T < \infty$,

$$\tilde{H}_2(N, K; T) = \lim_{N \to \infty} \frac{\sum_{j=1}^{\infty} \int_0^T [(\theta t)^j / j!]^e^{-\theta t} p_K(t)h(t)dt}{\sum_{j=1}^{\infty} \int_0^\infty [(\theta t)^j / j!]^e^{-\theta t} p_K(t)dt},$$

which increases strictly with $N$ from $H_2(\infty, K; T)$ to $h(T)$ and increases strictly with $K$ from $H_2(N, 0; T)$ to $h(T)$.

Proof. Note that

$$\tilde{H}_2(\infty, K; T) = \lim_{N \to \infty} \frac{\sum_{j=1}^{\infty} \int_0^T [(\theta t)^j / j!]^e^{-\theta t} p_K(t)h(t)dt}{\sum_{j=1}^{\infty} \int_0^\infty [(\theta t)^j / j!]^e^{-\theta t} p_K(t)dt} = h(T),$$

$$\tilde{H}_2(N, \infty; T) = \lim_{K \to \infty} \frac{\sum_{j=1}^{\infty} \int_0^T [(\theta t)^j / j!]^e^{-\theta t} p_K(t)h(t)dt}{\sum_{j=1}^{\infty} \int_0^\infty [(\theta t)^j / j!]^e^{-\theta t} p_K(t)dt} = h(T).$$

Thus, by using the similar method of Appendix B, we can prove Appendix D. Therefore, $\tilde{H}_2(N, K) \equiv H_2(N, K; \infty)$ increases strictly with $N$ from $1/\int_0^\infty p_K(t)dt$ to $h(\infty)$ and increases strictly with $K$ from $H_2(N, 0)$ to $h(\infty)$.

Appendix E:

For $N = 0, 1, 2, \ldots$ and $0 < T < \infty$,

$$Q_{O1}(N; T) \equiv \frac{\int_0^T (\theta t)^N e^{-\theta t} h(t)dt}{\int_0^T (\theta t)^N e^{-\theta t} h(t)\left[\int_0^\infty e^{-H(u)+H(t)}du\right]dt}$$

increases strictly with $N$ from $Q_{O1}(0; T)$ to $Q_{O1}(\infty; T) = F(T)/\int_T^\infty F(t)dt$.

Proof. Note that

$$Q_{O1}(\infty; T) = \lim_{N \to \infty} Q_{O1}(N; T) = \frac{1}{\int_T^\infty e^{-H(t)+H(t)}dt} = \frac{F(T)}{\int_T^\infty F(t)dt}.$$

Denoting

$$L_5(T) \equiv \int_0^T (\theta t)^N e^{-\theta t} h(t)dt \times \int_0^T (\theta t)^N e^{-\theta t} h(t)\left[\int_0^\infty e^{-H(u)+H(t)}du\right]dt$$

$$- \int_0^T (\theta t)^N e^{-\theta t} h(t)dt \times \int_0^T (\theta t)^N e^{-\theta t} h(t)\left[\int_0^\infty e^{-H(u)+H(t)}du\right]dt,$$

we have $L_5(0) = 0$ and

$$L_5'(T) = (\theta T)^N e^{-\theta T} h(T)\int_0^T (\theta T)^N e^{-\theta T} (\theta T - \theta)$$

$$\times \left[\int_T^\infty e^{-H(u)+H(t)}du - \int_T^\infty e^{-H(u)+H(t)}du\right]dt > 0,$$

which follows that $Q_{O1}(N; T)$ increases strictly with $N$ from $Q_{O1}(0; T)$ to $\tilde{F}(T)/\int_T^\infty \tilde{F}(t)dt$ for any $T$. Thus, because $T$ is arbitrary, $Q_{O1}(N) \equiv Q_{O1}(N; \infty)$ increases strictly with $N$ from $Q_{O1}(0)$ to $\lim_{T \to \infty} \tilde{F}(T)/\int_T^\infty \tilde{F}(t)dt = h(\infty)$.

Appendix F:

For $K = 0, 1, 2, \ldots$ and $0 < T < \infty$,

$$H_{O1}(K; T) \equiv \frac{\int_0^T h(t)^Ke^{-H(t)}\left[\int_0^\infty \theta e^{-\theta t} h(t + u)du\right]dt}{\int_T^\infty h(t)^Ke^{-H(t)}dt}$$

which increases strictly with $K$ from $H_{O1}(0; T)$ to $\int_0^\infty \theta e^{-\theta (T + t)}dt$.

Proof. Note that

$$H_{O1}(\infty; T) = \lim_{K \to \infty} H_{O1}(K; T) = \int_0^\infty \theta e^{-\theta (T + t)}dt,$$

Thus, by using the similar method of Appendix E, $H_{O1}(K; T)$ increases strictly with $K$ from $H_{O1}(0; T)$ to $\int_0^\infty \theta e^{-\theta h(T + t)}dt$ for any $T$. Therefore, because $T$ is arbitrary, $H_{O1}(K) \equiv H_{O1}(K; \infty)$ increases strictly with $K$ from $H_{O1}(0)$ to $\lim_{T \to \infty} \int_0^\infty \theta e^{-\theta (T + t)}dt = h(\infty)$.
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