On the Strength of Damping Effect in Online User Dynamics for Preventing Flaming Phenomena

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SUMMARY As online social networks (OSNs) have become remarkably active, we often experience explosive user dynamics such as online flaming, which can significantly impact the real world. Since the rapidity with which online user dynamics propagates, countermeasures based on social analyses of the individuals who cause online flaming take too much time that timely measures cannot be taken. To consider immediate solutions without individuals’ social analyses, a countermeasure technology for flaming phenomena based on the oscillation model, which describes online user dynamics, has been proposed. In this framework, the strength of damping to prevent online flaming was derived based on the wave equation of networks. However, the assumed damping strength was to be a constant independent of the frequency of user dynamics. Since damping strength may generally depend on frequency, it is necessary to consider such frequency dependence in user dynamics. In this paper, we derive the strength of damping required to prevent online flaming under the general condition that damping strength depends on the frequency of user dynamics. We also investigate the existence range of the Laplacian matrix’s eigenvalues representing the OSN structure assumed from the real data of OSNs, and apply it to the countermeasure technology for online flaming.

key words: online flaming, user dynamics, oscillation model, eigenvalue, Laplacian matrix

1. Introduction

In recent years, with the spread of social networking services (SNS) such as Twitter and Facebook, users’ activities in online social networks have come to be closely connected to online activities in the real world, not only in online communities. As a result, the effects of explosive online user dynamics, including online flaming phenomena, are becoming more serious, and countermeasures are needed [1], [2].

To resolve online flaming phenomena essentially, it is desirable to eliminate their causes by analyzing each event in detail, one by one. To realize this, it is necessary to understand the user dynamics on SNS, and for that purpose, many studies based on statistical analyses using actual SNS data have been performed. Online flaming or some kinds of such user dynamics have been investigated in YouTube [3], Twitter [4], [5], and WhatsApp [6]. [3] investigates events displaying hostility by using some offensive language in YouTube. [4] analyzes the relationship between hate speech and journalist-related accounts on Twitter. [5] analyzes cyberbullying and cyberaggression in Twitter to detect hate-related communities. [6] investigates the characteristics of the propagation of fake images in SNS.

The phenomenological analyses based on actual data may clarify each case-specific social structure, but it requires too much time to prevent the damage from spreading. In particular, fake news or negative information tends to propagate more rapidly than normal information in SNS [7], the countermeasures of online flaming needs time-sensitive operation. Thus we need a theoretical framework for flaming countermeasures that do not depend on the details of each individual event. Such a framework has been proposed based on the oscillation model on networks [8] which is used to describe online user dynamics [9], [10]. The oscillation model is a framework modeled by the wave equation on the network by assuming that the influence between users that affect the user dynamics is propagated through OSNs at a finite speed.

The flaming countermeasure based on the oscillation model can prevent explosive user dynamics including online flaming, by strengthening the damping effect of user dynamics on OSNs [11]. The occurrence of online flaming is related to the eigenvalues of the Laplacian matrix representing the OSN structure, and when online flaming occurs, the Laplacian matrix has non-real eigenvalues. The strength of the damping effect needed to prevent online flaming is determined by the distribution of the non-real eigenvalues of the Laplacian matrix. However, to know the eigenvalues of the Laplacian matrix representing the OSN, we need to fully understand the structure of the OSN, but it is impossible because the structure of the whole OSNs cannot be visualized. Therefore, it is necessary to consider the countermeasure without knowing the exact distribution of eigenvalues of the Laplacian matrix. In a previous study, the strength of the damping effect for preventing online flaming was derived under specific conditions [8]. However, there are the following issues:

- In general oscillating phenomena, the strength of the damping effect depends on frequency, but it was considered under the condition that the strength of the damping effect of user dynamics is a constant independent of the frequency.
- Since the Laplacian matrix representing the OSN is unknown, we assumed the largest theoretically possible range for the distribution of its eigenvalues.

In general, the strength of the damping effect may depend on the frequency, and the frequency dependence of the
damping characteristics of user dynamics has been theoretically clarified [12]. Therefore, the conditions for preventing online flaming should be extended for considering the frequency dependence.

Also, although we cannot observe the eigenvalue distribution of real OSNs, we may be able to find some characteristics regarding the range of their existence. Social networks, including OSNs, have been studied as a kind of complex network, and they have properties such as scale-free and small world [15], [16]. In the famous network model of complex networks, there is a relationship between the structure of the network and the distribution of eigenvalues of the Laplacian matrix, suggesting that the distribution of eigenvalues is characterized by the nodal degree distribution [17].

If the distribution of eigenvalues of the Laplacian matrix of the OSN is distributed within a specific range, it is possible to prevent online flaming without complete knowledge of eigenvalues of the Laplacian matrix. If the eigenvalues of OSNs are distributed within a narrower range compared to the entire existence range of eigenvalues, it is expected that a weaker damping effect will be used to prevent online flaming than that required by the conventional countermeasure technology.

In this paper, we consider the case that the strength of the damping effect of the user dynamics depends on the frequency, and derive the condition of the strength of the damping effect to prevent online flaming. We also investigate the existence range of eigenvalues of Laplacian matrices assumed from real data of OSNs and apply them to countermeasures for online flaming.

2. Oscillation Model for Describing Online User Dynamics

Let us consider the directed graph $G(V, E)$ with $n$ nodes as the structure of OSNs, where $V = \{1, \ldots, n\}$ is the set of nodes and $E$ is the set of directed links. The directed from node $i$ to node $j$ is expressed by $(i \to j) \in E$ and $w_{ij} (\geq 0)$ is its link weight. Then the adjacency matrix $\mathcal{A} = [\mathcal{A}_{ij}]_{1 \leq i,j \leq n}$ is defined as follows:

$$\mathcal{A}_{ij} := \begin{cases} w_{ij} & (i \to j) \in E, \\ 0 & (i \to j) \notin E, \end{cases}$$

Next, we define the weighted out-degree $d_i := \sum_{j \in \partial i} w_{ij}$ of node $i$ and degree matrix as follows:

$$\mathcal{D} := \text{diag}(d_1, \ldots, d_n),$$

where $\partial i$ denotes the set of nodes adjacent to node $i$. Then, we define the Laplacian matrix $\mathcal{L}$ as follows:

$$\mathcal{L} := \mathcal{D} - \mathcal{A}.$$

In addition, let the eigenvalues of $\mathcal{L}$ be $\lambda_{\mu}$ ($\mu = 0, 1, \ldots, n - 1$) and the eigenvectors associated with $\lambda_{\mu}$ by $\mathbf{v}_{\mu}$. We assume the eigenvalues are not duplicated.

The eigenvalues $\lambda_{\mu}$ are generally complex numbers, whose range of existence is given by the largest Gershgorin disk [18] (Fig.1) of $\mathcal{L}$ as

$$D_{\max}^G(\mathcal{L}) = \{ z \in \mathbb{C} : |z - d_{\max}| \leq d_{\max} \},$$

where $d_{\max}$ is the maximum weighted out-degree of the network. It is known that all the eigenvalues of $\mathcal{L}$ lie within its largest Gershgorin disk [8].

The oscillation model [8], [9] is a minimal model for describing user dynamics in OSNs. Let $x_i(t)$ be the state of node (user) $i$ at time $t$. Since the influence of interaction between users must propagate through any OSN at a finite speed, its description by the wave equation should be possible, which is the equation for describing the propagation of something at finite speed.

In the oscillation model, for each adjacent node pair, $i$ and $j$, the force acting at node $i$ from node $j$ is given as

$$-w_{ij} (x_i(t) - x_j(t)).$$

For the state vector $x(t) := (x_1(t), \ldots, x_n(t))$, the wave equation in the OSN is written as

$$\frac{d^2}{dt^2} x(t) + \mathbf{\Gamma} \frac{dx(t)}{dt} = -\mathcal{L} x(t),$$

where $\mathbf{\Gamma}$ is the matrix expressing the strength of the damping. This equation describes how the influence between users propagates.

Here, let us consider the frequency dependence of the strength of damping according to the discussion in [12]. First, we assume $\mathbf{\Gamma} := \gamma \mathbf{I}$, where $\gamma$ is a non-negative constant and $\mathbf{I}$ is the unit matrix. By substituting the expansion of $x(t)$ by $\mathbf{v}_{\mu}$ as

$$x(t) = \sum_{\mu=0}^{n-1} a_{\mu}(t) \mathbf{v}_{\mu},$$

into the wave equation (3), we have the equation of motion for each oscillation mode $a_{\mu}(t)$ ($\mu = 0, 1, \ldots, n - 1$) as

$$\frac{d^2}{dt^2} a_{\mu}(t) + \gamma \frac{da_{\mu}(t)}{dt} a_{\mu}(t) = -\lambda_{\mu} a_{\mu}(t).$$

Since the damping coefficient $\gamma$ is the same for all the oscillation modes, this expresses frequency-independent damping.
where \( n \) is the damping coefficient \( \gamma \) with a frequency-dependent function \( \gamma(\omega_\mu) \), where \( \omega_\mu := \sqrt{A_\mu} \). The replacement yields

\[
\frac{d^2}{dr^2} a_\mu(t) + \gamma(\omega_\mu) \frac{d}{dt} a_\mu(t) = -\lambda_\mu a_\mu(t).
\] (6)

This equation includes frequency-dependent damping effects. Next, let us return (6) to the form of (3) and consider the form of the matrix \( \Gamma \). \( \Gamma \) should only have non-diagonal components corresponding to links of OSN. This is because the damping force should act only between adjacent node pairs, and any direct effect does not act between node pairs without direct links. In addition, the rule of damping should be the same for any structure of OSN. From these two conditions, we have

\[
\Gamma = \gamma_0 I + \gamma_1 L,
\]

and

\[
\gamma(\omega_\mu) := \gamma_0 + \gamma_1 \lambda_\mu,
\]

where \( \gamma_0 \) and \( \gamma_1 \) are constants [12]. Note that \( \text{Re}[\gamma(\omega_\mu)] = \gamma_0 + \gamma_1 \text{Re}[\lambda_\mu] \geq 0 \).

The solution of (6) is written as

\[
a_\mu(t) = c_\mu^+ \exp \left[ -\frac{\gamma(\omega_\mu)}{2} t + i \sqrt{r_\mu} \exp \left( i \frac{\theta_\mu}{2} \right) \right] + c_\mu^- \exp \left[ -\frac{\gamma(\omega_\mu)}{2} t - i \sqrt{r_\mu} \exp \left( i \frac{\theta_\mu}{2} \right) \right],
\] (7)

where \( c_\mu^+ \) and \( c_\mu^- \) are constants that depend on \( \mu \) and \( r_\mu \) and \( \theta_\mu(-\pi < \theta_\mu \leq \pi) \) are, respectively, the absolute value and the argument of the following complex number:

\[
r_\mu \exp(i \theta_\mu) := \lambda_\mu - \left( \frac{\gamma(\omega_\mu)}{2} \right)^2.
\] (8)

In the oscillation model, the oscillation energy can be considered as the strength of the activity of user dynamics [19],[20]. Also, the situation in which oscillation energy diverges over time is considered to describe explosive user dynamics, which include online flaming phenomena. Therefore, in order to prevent explosive user dynamics, it is necessary to consider the conditions under which the oscillation energy does not diverge. By deriving the strength of the damping that satisfies this condition, we can obtain a framework in which the damping effect can prevent online flaming phenomena.

3. Technology to Counter Online Flaming Based on the Oscillation Model

3.1 Model of Explosive User Dynamics Considering Frequency-Dependent Damping Coefficient

Although individual nodes in the oscillation model seem complex behaviors, (6) means that \( n \) types of oscillation modes appear throughout the whole network, and each of them behaves like a harmonic oscillator. Since the oscillation energy of the harmonic oscillator with a unit mass, the angular frequency \( \omega \), and the amplitude \( A \) is determined by \( \frac{1}{2} \omega^2 A^2 \), the oscillation energy of each oscillation mode \( \mu \) is given by \( \frac{1}{2} \omega_\mu^2 [a_\mu(t)]^2 \).

Since the oscillation energy is proportional to the square of the absolute value of \( a_\mu(t) \), we derive the condition that \( a_\mu(t) \) does not diverge as the condition that online flaming does not occur.

By decomposing the damping coefficient \( \gamma(\omega_\mu) \) into real and imaginary parts as in

\[
\gamma(\omega_\mu) = (\gamma_0 + \gamma_1 \text{Re}[\lambda_\mu]) + i \gamma_1 \text{Im}[\lambda_\mu].
\] (9)

(7) is rewritten as

\[
a_\mu(t) = c_\mu^+ \exp \left[ -\frac{\gamma_0 + \gamma_1 \text{Re}[\lambda_\mu]}{2} t - \sqrt{r_\mu} \sin \left( \frac{\theta_\mu}{2} \right) \right] \\
\times c_\mu^* \exp \left[ -\frac{\gamma_0 + \gamma_1 \text{Re}[\lambda_\mu]}{2} t + \sqrt{r_\mu} \sin \left( \frac{\theta_\mu}{2} \right) \right] \\
+ c_\mu^- \exp \left[ -\frac{\gamma_0 + \gamma_1 \text{Re}[\lambda_\mu]}{2} t + \sqrt{r_\mu} \sin \left( \frac{\theta_\mu}{2} \right) \right] \\
\times c_\mu^* \exp \left[ -\frac{\gamma_0 + \gamma_1 \text{Re}[\lambda_\mu]}{2} t - \sqrt{r_\mu} \sin \left( \frac{\theta_\mu}{2} \right) \right].
\] (10)

To determine whether \( a_\mu(t) \) diverges or not, we need to check whether the real components of the exponent of the exponential function in (10) are positive or negative. If \( a_\mu(t) \) diverges, the following condition is satisfied:

\[
\left( \frac{\text{Re}[\gamma(\omega_\mu)]}{2 \sqrt{r_\mu}} + \sin \left( \frac{\theta_\mu}{2} \right) \right) \left( \frac{\text{Re}[\gamma(\omega_\mu)]}{2 \sqrt{r_\mu}} - \sin \left( \frac{\theta_\mu}{2} \right) \right) < 0.
\] (11)

This means one of the real components of the exponent is positive and the other is negative.

Consequently, the condition under which the oscillation energy diverges is given by

\[
\text{Re}[\gamma(\omega_\mu)] < \sin \left( \frac{\theta_\mu}{2} \right),
\] (12)

and the condition that the oscillation energy does not diverge is obtained as

\[
\text{Re}[\gamma(\omega_\mu)] \geq \sin \left( \frac{\theta_\mu}{2} \right).
\] (13)

3.2 Countermeasure to Flaming Phenomena Given the Frequency-Dependent Damping Coefficient

Based on condition (13), i.e., the oscillation energy does not diverge, we derive the condition of damping coefficient to satisfy (13).

In parameters \( \gamma_0 \) and \( \gamma_1 \), which determine the strength
of damping, the strength of frequency dependence is determined by \( \gamma_1 \). Since we cannot control the frequency dependency of the damping effect, the value of \( \gamma_1 \) is considered as a parameter predetermined by the characteristics of OSNs. Therefore, \( \gamma_0 \) is the only parameter that can be manipulated independently of the network structure. In this framework, for any given \( \gamma_1 \), we try to set an appropriate value of \( \gamma_0 \) to prevent the flaming phenomena. Here, the actual action to increase the value of \( \gamma_0 \) includes disseminating other information to attract users’ attention. In the following, we derive the value of \( \gamma_0 \) necessary to prevent the triggering of explosive user dynamics.

The distributed range of eigenvalues of \( L \) is the interior of the largest Gershgorin disk (including its boundary) of radius \( d_{\text{max}} \) with center \((d_{\text{max}},0)\). From condition (13), we can determine the range satisfying condition (13) on the complex plane. So, if the largest Gershgorin disk of \( L \) lies completely within the range satisfying condition (13) on the complex plane, the oscillation energy never diverges regardless of details of the network structure.

To find out what region satisfying the condition (13) in the complex plane, where the oscillation energy does not diverge, is drawn, we transform inequality (13). By using \( \text{Re}[\lambda] \) and \( \text{Im}[\lambda], \) (8) is expressed as

\[
\text{Re}[\lambda] \exp(i\theta) = X + \frac{1}{4} \gamma_1^2 \text{Im}[\lambda]^2 + iY,
\]

where
\[
X = \text{Re}[\lambda] - \frac{1}{4} \left( \gamma_0^2 + 2 \gamma_0 \gamma_1 \text{Re}[\lambda] + \gamma_1^2 \text{Re}[\lambda]^2 \right),
\]
\[
Y = \text{Im}[\lambda] - \frac{1}{2} \left( \gamma_0 \gamma_1 \text{Im}[\lambda] + \gamma_1^2 \text{Re}[\lambda] \text{Im}[\lambda] \right).
\]

By using Euler’s formula,
\[
\exp(i\theta) = \cos \theta + i \sin \theta,
\]

\( r_{\mu} \) and \( \cos \theta_{\mu} \) is expressed as follows:
\[
r_{\mu} = \sqrt{\left( X + \frac{1}{4} \gamma_1^2 \text{Im}[\lambda]^2 \right)^2 + Y^2},
\]
\[
\cos \theta_{\mu} = \frac{1}{r_{\mu}} \left( X + \frac{1}{4} \gamma_1^2 \text{Im}[\lambda]^2 \right).
\]

Since the both side of (13) are positive, by squaring both sides of inequality (13), we have
\[
\left( \frac{\gamma_0 + \gamma_1 \text{Re}[\lambda]}{2 \sqrt{\mu}} \right)^2 \geq \sin^2 \frac{\theta_{\mu}}{2} = \frac{1 - \cos \theta_{\mu}}{2}.
\]

Therefore, we have
\[
(\gamma_0 + \gamma_1 \text{Re}[\lambda])^2 + 2r_{\mu} \cos \theta_{\mu} \geq 2r_{\mu}.
\]

Since the both side of (18) also are positive, by squaring both sides of inequality (18), we have
\[
\left( (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2 + 2r_{\mu} \cos \theta_{\mu} \right) \geq 4 r_{\mu}^2.
\]

By expanding the inequality (19), the condition that the oscillation energy does not diverge can be expressed by using \( \text{Re}[\lambda] \) and \( \text{Im}[\lambda] \) as follows (see Appendix A):
\[
(1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda]) \text{Im}[\lambda]^2 \leq \text{Re}[\lambda] (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2.
\]

Let us consider the following two cases:

• if \( 1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda] > 0, \)
\[
\text{Im}[\lambda]^2 \leq \frac{\text{Re}[\lambda] (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2}{1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda]}.
\]

• if \( 1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda] \leq 0, \)
\[
\text{Im}[\lambda]^2 \geq \frac{\text{Re}[\lambda] (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2}{1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda]}.
\]

In the latter case, the right-hand side of (22) is always negative since \( 0 \leq \text{Re}[\lambda] \leq 2 d_{\text{max}} \), and the inequality always holds. So the condition of no divergence of oscillation energy always holds. Therefore, we only need to consider the condition (21) for the case \( 1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda] > 0. \)

The region of eigenvalue distribution of \( L \), determined by the Gershgorin theorem, are written as
\[
\text{Im}[\lambda]^2 \leq d_{\text{max}}^2 - (\text{Re}[\lambda] - d_{\text{max}})^2.
\]

We compare (23) and the condition (21) that the oscillation energy does not diverge. The condition that the largest Gershgorin disk is completely included the range of (21) is expressed as
\[
\frac{d_{\text{max}}^2 - (\text{Re}[\lambda] - d_{\text{max}})^2}{\text{Re}[\lambda] (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2} \leq \frac{\text{Re}[\lambda] (\gamma_0 + \gamma_1 \text{Re}[\lambda])^2}{1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda]}.
\]

From \( 1 - \gamma_0 \gamma_1 - \gamma_1^2 \text{Re}[\lambda] > 0 \) and \( \text{Re}[\lambda] \geq 0, \) (24) can be rewritten as follows:
\[
(\gamma_0 \gamma_1 + 1 + 2 \gamma_1^2 d_{\text{max}}) \text{Re}[\lambda] \geq - (\gamma_0^2 + 2 \gamma_0 \gamma_1 d_{\text{max}} - 2 d_{\text{max}}).
\]

We consider the conditions for satisfying inequality (25) in the following three cases.

• if \( \gamma_0 \gamma_1 + 1 + 2 \gamma_1^2 d_{\text{max}} = 0, \)
\[
\gamma_0^2 + 2 \gamma_0 \gamma_1 d_{\text{max}} - 2 d_{\text{max}} \geq 0.
\]

Since \( \gamma_0 \geq 0, \) the range of \( \gamma_0 \) is
\[
\gamma_0 \geq \sqrt{\gamma_1^2 d_{\text{max}}^2 + 2 d_{\text{max}} - \gamma_1 d_{\text{max}} - 2 d_{\text{max}}}
\]

• if \( \gamma_0 \gamma_1 + 1 + 2 \gamma_1^2 d_{\text{max}} > 0, \)
\[ \gamma_0^2 + 2 \gamma_0 \gamma_1 d_{\text{max}} - 2 d_{\text{max}} \geq -\text{Re}[\lambda_i]. \]  
(28)

Since \( \min(\text{Re}[\lambda_i]) = 0 \), in order for inequality (28) to hold for all eigenvalues, the following inequality must be satisfied.

\[ \gamma_0^2 + 2 \gamma_0 \gamma_1 d_{\text{max}} - 2 d_{\text{max}} \geq 0. \]  
(29)

Since \( \gamma_0 \geq 0 \), the condition of \( \gamma_0 \) to ensure the non-divergence of oscillation energy for all eigenvalues is written as

\[ \gamma_0 \geq \sqrt{\gamma_1^2 d_{\text{max}}^2 + 2 d_{\text{max}} - \gamma_1 d_{\text{max}}}. \]  
(30)

- if \( \gamma_0 \gamma_1 + 1 + 2 \gamma_1^2 d_{\text{max}} < 0 \).

\[ \gamma_0^2 + 2 \gamma_0 \gamma_1 d_{\text{max}} - 2 d_{\text{max}} \geq -\text{Re}[\lambda_i]. \]  
(31)

Since \( \max(\text{Re}[\lambda_i]) = 2 d_{\text{max}} \), in order for inequality (31) to hold for all eigenvalues, (31) is rewritten as

\[ (\gamma_0 + 2 \gamma_1 d_{\text{max}})^2 \geq 0. \]  
(32)

Therefore, inequality (31) always holds in this case.

To summarize the above results, the condition for \( \gamma_0 \) that ensures the oscillation energy does not diverge for all eigenvalues is obtained as

\[ \gamma_0 \geq \sqrt{\gamma_1^2 d_{\text{max}}^2 + 2 d_{\text{max}} - \gamma_1 d_{\text{max}}}. \]  
(33)

Therefore, for given the maximum weighted out-degree of the network, \( d_{\text{max}} \), and parameter \( \gamma_1 \) of the damping coefficient, we can prevent flaming phenomena by setting the value of \( \gamma_0 \) to satisfy (33). Note that we do not necessarily adjust the value of \( \gamma_0 \) to the minimum value satisfying (33). Since the values of \( \gamma_1 \) and \( d_{\text{max}} \) are difficult to determine, it is probably impossible to such an adjustment. What is important in the result is that the fact that increasing the damping strength \( \gamma_0 \) can stop divergence regardless of the values of \( \gamma_1 \) or \( d_{\text{max}} \) is theoretically guaranteed. Furthermore, it is important that the required value of \( \gamma_0 \) is independent of details of the topology structure of the network and is determined by the maximum node degree \( d_{\text{max}} \). If there is no frequency dependence of the damping coefficient, that is \( \gamma_1 = 0 \), so the condition 33 is reduced as follows:

\[ \gamma_0 \geq \sqrt{2 d_{\text{max}}}. \]

This is consistent with the conditions for the damping coefficient without frequency dependence, derived in [8].

We show the examples of complete prevention of online flaming with \( \gamma_0 \geq \sqrt{\gamma_1^2 d_{\text{max}}^2 + 2 d_{\text{max}} - \gamma_1 d_{\text{max}}} \). Where \( \gamma_0 \) satisfies condition (33). Figure 2 shows the regions in which the oscillation energy does not diverge as given by condition (13), for the cases of \( \gamma_1 > 0 \), \( \gamma_1 = 0 \), and \( \gamma_1 < 0 \). These regions are depicted in blue. In addition, the largest Gershgorin disk of \( \mathcal{L} \) is depicted in red. In all figures, it can be seen that the regions wherein the oscillation energy does not diverge completely include the largest Gershgorin disk so that no divergence of oscillation energy occurs regardless of the details of the network structure.

Next, we show the examples of incomplete prevention of online flaming by using \( \gamma_0 < \sqrt{\gamma_1^2 d_{\text{max}}^2 + 2 d_{\text{max}} - \gamma_1 d_{\text{max}}} \), in which \( \gamma_0 \) is less than the minimum value that satisfies condition (33). Figure 3 shows the regions wherein the oscillation energy does not diverge as indicated by condition (13), for the cases of \( \gamma_1 > 0 \), \( \gamma_1 = 0 \), and \( \gamma_1 =< 0 \). In these cases, the regions cannot completely enclose the Gershgorin disk. If even just one eigenvalue appears outside of the region, the oscillation energy diverges and the flaming phenomenon occurs. Therefore, depending on the position of the eigenvalues of \( \mathcal{L} \), flaming prevention is not assured.

4. Evaluation of Eigenvalue Distribution for Laplacian Matrices of Online Social Networks

In the countermeasure technology proposed in the previous section, the condition (21) of the strength of the damping effect required to prevent online flaming is designed with the policy of ensuring the inclusion of all the eigenvalues of the Laplacian matrix within the region determined by the condition (13). This is the result of considering the distribution range of the eigenvalues of the Laplacian matrix of the OSN as the largest theoretically possible region since we cannot actually know the details of the structure of the OSN. However, there may be some bias in the distribution range of the eigenvalues of the Laplacian matrix of the real OSN. If the distribution of eigenvalues of the Laplacian matrix is biased, it is no longer necessary to enclose the entire Gershgorin disk with the region determined by the condition (13), and we can weaken the damping effect for preventing online flaming.

In this section, we simulate the assigning of link weight with various patterns of nodal-degree dependent random numbers using real data of OSNs and conduct experiments to evaluate the eigenvalue distribution of the Laplacian matrix of the generated network.

4.1 Evaluation of Eigenvalue Distribution Based on Real Data

In this experiment, we evaluate the eigenvalue distribution of the Laplacian matrix of OSNs using real OSN data. The data for this experiment is link structure on the Facebook (friend relation on the Facebook) provided as an open dataset [21]. We use the dataset of Howard-90 (Facebook Networks) \((n = 4047)\). Since the data is observed as an unweighted undirected graph, it is necessary to suppose experimental conditions for the direction and weights of the links that reflect the various possibilities that can be assumed.

First, we consider the links given as an undirected graph as directed links, and the weights of the bidirectional links are given separately. The weight of a link is assigned by a
random variable that follows a probability distribution depending on the nodal degree of the nodes connected to the link, and two patterns are considered: one where the weight is assigned to the incoming link to the node, and the other where the weight is assigned to the outgoing link from the node. The probability distribution of the link weights is given by certain distributions with the mean being a power of the nodal degree, and several patterns are considered for the value of the power of the nodal degree. These patterns include a case of dependence on the 0-th power of nodal degree, that is, this pattern corresponds that the link weights and nodal degrees are independent. Since the structure of the actual online social network does not know what weights are given to the links, this paper evaluates the relationship in which the weights of the links depend on the node degree in various patterns to consider the wider possibility.

We denote that the probability distribution followed by the link weights by a probability density function $P(m; x)$ with mean $m$. As an example, we show a procedure to determine the weights of incoming and outgoing links by using a probability distribution whose mean is the power of the nodal degree. Figs. 4 and 5 show how the weights of the incoming and outgoing links of node $2$ are determined by the probability density function $P(d_2^a; x)$ with the average $d_2^a$, where $d_2$ denotes the nodal degree of node 2.

We consider two types of probability distributions for assigning the link weights: the exponential distribution and the Pareto distribution with the shape parameter set to 2. The probability density function of exponential distribution with mean $m$ is given as

$$ w_{12} \sim P(d_2^a; x) $$

and the probability density function of Pareto distribution with mean $m$ and the shape parameter set to 2 is given as

$$ P(m; x) = \frac{1}{m} \exp \left( \frac{1}{m x} \right). $$

To summarize the above, the patterns of the random
Table 1  Pattern of the random number for link weight

<table>
<thead>
<tr>
<th>Item</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution</td>
<td>exponential distribution, Pareto distribution</td>
</tr>
<tr>
<td>Mean</td>
<td>the $a$-th power of the nodal degree ($-1 \leq a \leq 2$)</td>
</tr>
<tr>
<td>Direction</td>
<td>incoming, outgoing</td>
</tr>
</tbody>
</table>

variable to assign the link weights are simulated by a combination of three items: the type of distribution of the random variables, the mean of the random variables that depend on the nodal degree, and the directions of the links (see Tab. 1).

The distribution of the eigenvalues is evaluated by the ellipse with the smallest area that includes all the regions where the non-real eigenvalues exist (the minimum inclusion ellipse). Note that real eigenvalues do not cause explosive user dynamics because they always satisfy the equation (13). Therefore, only eigenvalues that are not real numbers are subject to eigenvalue distribution evaluation. An example of this is shown in Fig. 6. We evaluate the eigenvalue distribution of OSNs within the largest Gershgorin disk of the Laplacian matrix of OSNs by evaluating the ranges spread to the real and imaginary axial directions: that is the major axis and the minor axis of the minimum inclusion ellipse. Note that the size of the largest Gershgorin disk of the Laplacian matrix is determined by the largest weighted out-degree $d_{\text{max}}$ in the network, so the minor axis and major axis are evaluated as normalized values by dividing by $d_{\text{max}}$.

4.2 Evaluation Results of Eigenvalue Distributions

In this section, we show the results of eigenvalue distributions by using the minimum inclusion ellipse.

First, we show some experimental results of the minimum inclusion ellipse in Figs. 7 – 10. It can be seen that the expansion in the imaginary direction is much narrower than the expansion in the real direction. In addition, these ranges are very narrow comparing the largest Gershgorin disk. The same feature was observed in all the experimental patterns.

Next, we show the change in the normalized minor axis and major axis with respect to the change in the mean of the random variable for the link weights. The change in normalized minor axis is shown in Fig. 11 and the change in normalized major axis is shown in Fig. 12. In both figures, the horizontal axis shows the power of the nodal degree, which determines the mean of the probability distribution for assigning the link weights.
nodal degree. If the mean of the random variables for assigning the link weights has an exponent that is less than the $-1$ power of the nodal degree, the above reversal phenomenon occurs. From the above consideration, it is sufficient to evaluate the mean of the random numbers that the link weights follow by an exponent greater than the $-1$-th power of the nodal degree.

From Figs. 11 and 12, the maximum minor axis of the minimum inclusion ellipse is about $2\%$ of $d_{\text{max}}$, and a maximum major axis of the minimum inclusion ellipse is about $30\%$ of $d_{\text{max}}$. So, the distribution is much narrower than that of the largest Gershgorin disk of the Laplacian matrix (disk with center and radius $d_{\text{max}}$). Therefore, it can be recognized that it is possible to prevent online flaming with a weaker damping effect than that required by the condition designed with the policy of enclosing the entire largest Gershgorin disk of the Laplacian matrix (21).

Since the distribution of eigenvalues is widest in the case where the mean of the random variable for assigning the link weights is between $-1$-th power and $0$-th power of the nodal degree, we can design countermeasures of online flaming based on these conditions as the worst cases. This indicates that, without considering the actual link weight of OSNs, which is difficult to observe numerically, we can expect to apply countermeasures for online flaming based only on the presence of links.

5. Conclusion

In this paper, we clarified the condition to prevent online flaming and evaluated the eigenvalue distribution of OSNs based on the oscillation model. Based on the oscillation model, all the eigenvalues of the Laplacian matrix describing...
OSNs are required to be distributed within a certain region on the complex plane. First, considering the possibility that the damping characteristics of user dynamics depend on the frequency of user dynamics, we derived a general condition to prevent occurring explosive user dynamics. Next, we evaluated the eigenvalue distribution of real OSNs by using the real data of link structure of friend relation on the Facebook as OSNs and assigning the link weights with various random variables. As a result, the distribution of non-real eigenvalues was found to be distributed in an area of about 0.6% of the original largest Gershgorin disk of the Laplacian matrix. There is a relationship between the area size of the eigenvalue distribution and the dependence on the nodal degree of the random variables for assigning link weights, and it was confirmed that the eigenvalues tend to be most widely distributed in cases where the mean of the random variables for assigning link weights is between $-1$-th and 0-th power of the nodal degree. Furthermore, since the area of non-real eigenvalues is about a very narrower area than the largest Gershgorin disk of the Laplacian matrix, it is sufficient to prevent explosive user dynamics by using a weaker damping effect than that required by conventional countermeasure technology designed to enclose the entire largest Gershgorin disk of the Laplacian matrix.

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References


Appendix A: Derivation of inequality (20)

The left-hand side of inequality (19) is written as

\[
\left( (\gamma_0 + \gamma_1 \Re[A_\mu])^2 + 2 \mu_\mu \cos \theta_\mu \right)^2
\]

\[
= \left( (\gamma_0 + \gamma_1 \Re[A_\mu])^2 + 2 \left( X + \frac{1}{4} \gamma_1^2 \Im[A_\mu]^2 \right) \right)^2
\]

\[
= (\gamma_0 + \gamma_1 \Re[A_\mu])^4 + 4 (\gamma_0 + \gamma_1 \Re[A_\mu])^2 X 
+ (\gamma_0 + \gamma_1 \Re[A_\mu])^2 \gamma_1^2 \Im[A_\mu]^2 
+ 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[A_\mu]^2 \right)^2.
\]

By using

\[
X = \Re[A_\mu] - \frac{1}{4} (\gamma_0 + \gamma_1 \Re[A_\mu])^2
\]

obtained from the definition of $X$, the left-hand side of inequality (19) is rewritten as

\[
\left( (\gamma_0 + \gamma_1 \Re[A_\mu])^2 + 2 \mu_\mu \cos \theta_\mu \right)^2
\]

\[
= \left( (\gamma_0 + \gamma_1 \Re[A_\mu])^2 + 2 \left( X + \frac{1}{4} \gamma_1^2 \Im[A_\mu]^2 \right) \right)^2
\]

\[
= (\gamma_0 + \gamma_1 \Re[A_\mu])^4 + 4 (\gamma_0 + \gamma_1 \Re[A_\mu])^2 X 
+ (\gamma_0 + \gamma_1 \Re[A_\mu])^2 \gamma_1^2 \Im[A_\mu]^2 
+ 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[A_\mu]^2 \right)^2.
\]
\[
\left( (\gamma_0 + \gamma_1 \Re[\lambda_{\mu}])^2 + 2 r_{\mu} \cos \theta_{\mu} \right)^2 \\
= 4 \left( (\gamma_0 + \gamma_1 \Re[\lambda_{\mu}])^2 \Re[\lambda_{\mu}] \right) \\
+ (\gamma_0 + \gamma_1 \Re[\lambda_{\mu}])^2 \gamma_1^2 \Im[\lambda_{\mu}]^2 \\
+ 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[\lambda_{\mu}]^2 \right)^2 .
\]  
\tag{A·1}

On the other hand, the right-hand side of inequality (19) is written as
\[
4 r_{\mu}^2 = 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[\lambda_{\mu}]^2 \right)^2 + 4 Y^2 \\
= 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[\lambda_{\mu}]^2 \right)^2 \\
+ 4 \Im[\lambda_{\mu}]^2 \left( 1 - \frac{1}{2} (\gamma_0 \gamma_1 + \gamma_1^2 \Re[\lambda_{\mu}]) \right)^2 \\
= 4 \left( X + \frac{1}{4} \gamma_1^2 \Im[\lambda_{\mu}]^2 \right)^2 \\
+ 4 \Im[\lambda_{\mu}]^2 \left( 1 - \gamma_0 \gamma_1 - \gamma_1^2 \Re[\lambda_{\mu}] \right) \\
+ \gamma_1^2 \Im[\lambda_{\mu}]^2 \left( \gamma_0 + \gamma_1 \Re[\lambda_{\mu}] \right)^2 .
\]  
\tag{A·2}

From (A·1) and (A·2), inequality (20) is derived.

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