ORTHOGONAL VARIABLE SPREADING FACTOR CODES OVER FINITE FIELDS∗∗∗

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SUMMARY The present paper proposes orthogonal variable spreading factor codes over finite fields for multi-rate communications. The proposed codes have layered structures that combine sequences generated by discrete Fourier transforms over finite fields, and have various code lengths. The design method for the proposed codes and examples of the codes are shown.

KEY WORDS: orthogonal variable spreading factor codes, multi-rate communications, finite fields, discrete Fourier transform

1. Introduction

Orthogonal spreading codes have been applied to wireless communication systems using direct sequence code division multiple access (DS-CDMA) [1]-[3]. Various orthogonal codes have been studied. Walsh codes that are derived from Hadamard matrices are one example of such codes [1]. Polyphase orthogonal codes over the complex field have been studied [4]. Recently, Hadamard-type matrices on finite fields and complete complementary codes have been derived by introducing the concept of the GF-conjugate [5].

Tree-structured generation of orthogonal spreading codes with different code lengths has been proposed, and these codes, called orthogonal variable spreading factor (OVSF) codes, have been used in DS-CDMA systems realizing multi-rate communications, which support a wide data rate range [6]-[9]. These OVSF codes are based on Walsh codes and are binary codes. The authors have proposed non-binary OVSF codes over the complex field [10], [11]. The tree structure combining polyphase orthogonal codes realizes the codes, and the binary OVSF codes are represented as a special case of non-binary OVSF codes.

The present study proposes OVSF codes over finite fields. The discrete Fourier transform (DFT) exists over an arbitrary field [12]. Row or column components of the DFT matrix yield the orthogonal sequences. The tree structure combining the sequences generates OVSF codes over finite fields when the DFT matrix is designed over prime fields or extension fields. The design method for the proposed codes is shown and examples of such codes are demonstrated.

The remainder of the present paper is organized as follows. The conventional OVSF codes are shown in Section 2, the properties of the DFT are shown in Section 3, and the proposed OVSF codes over finite fields are shown in Section 4.

2. Conventional OVSF codes

2.1 Binary OVSF codes

OVSF codes have been used in DS-CDMA wireless systems that support multi-rate communication services [6]-[9]. These codes are binary codes having elements in {±1, -1}. The two-point Hadamard matrix \( S_H^{(2)} \) is shown as

\[
S_H^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} s_H^{(2)0} \\ s_H^{(2)1} \end{bmatrix}^T
\]

where \( s_H^{(2)0} = [1 \ 1] \) and \( s_H^{(2)1} = [1 \ -1] \). \( s_H^{(2)0} \) corresponds to a Walsh code sequence of length 2 for \( i = 0, 1 \).

We define \( r_H^{(2)h} \) which is equal to \( s_H^{(2)h} \) for \( h = 0, 1 \). The inner product of \( s_H^{(2)} \) and \( r_H^{(2)} \) yields

\[
s_H^{(2)h} r_H^{(2)d} = \begin{cases} 2 & (h = i) \\ 0 & \text{otherwise} \end{cases}
\]

then \( s_H^{(2)} \) and \( r_H^{(2)} \) are orthogonal to each other for \( h \neq i \).

We define an \( M_1 \times M_1 \) square matrix \( X \) and an \( M_2 \times M_2 \) square matrix \( Y \) respectively, then the Kronecker product [1] of \( X \) and \( Y \) generates the \( M_1 M_2 \times M_1 M_2 \) matrix \( Z \) as

\[
Z = X \otimes Y
\]

\[
= \begin{bmatrix} x_{0,0} Y & x_{0,1} Y & \cdots & x_{0,M_1-1} Y \\ x_{1,0} Y & x_{1,1} Y & \cdots & x_{1,M_1-1} Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{M_1-1,0} Y & x_{M_1-1,1} Y & \cdots & x_{M_1-1,M_1-1} Y \end{bmatrix},
\]

where

\[
X = \begin{bmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,M_1-1} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,M_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{M_1-1,0} & x_{M_1-1,1} & \cdots & x_{M_1-1,M_1-1} \end{bmatrix}.
\]

The Kronecker product of \( S_H^{(2)} \) and \( S_H^{(2)} \) generates \( S_H^{(2,2)} \) as

\[
S_H^{(2,2)} = S_H^{(2)} \otimes S_H^{(2)} = \begin{bmatrix} S_H^{(2)} & S_H^{(2)} \\ S_H^{(2)} & -S_H^{(2)} \end{bmatrix}
\]
where $n$ is a positive integer. The complex number $\omega$ is a primitive $n$th root of unity and is represented by

$$\omega = e^{-j\pi/n},$$

where $j = \sqrt{-1}$. The $n$ different $n$th roots of unity are denoted as $\omega_0, \omega_1, \omega_2, \ldots, \omega^{n-1}$. Polyphase orthogonal code sequences of length $n$ are presented as the rows of $S_{C,n}$, which are defined as $S_{C,n,i} = \begin{bmatrix} \omega_0 & \omega_1 & \ldots & \omega^{(n-1)i} \end{bmatrix}$ for $i = 0, 1, \ldots, n-1$.

We define $r_{C,n,h}$ which is equal to $S_{C,n,h}$ for $h = 0, 1, \ldots, n-1$. Multiplying $S_{C,n,i}$ and $r_{C,n,h}$, where $r_{C,n,h}$ corresponds to a conjugate transpose of $r_{C,n,h}$, yields

$$S_{C,n,i} r_{C,n,h}^H = \begin{cases} n & (h = i), \\ 0 & \text{(otherwise)}. \end{cases}$$

For $n = 2$, $S_{C,2}^{(2)}$ is shown as

$$S_{C,2}^{(2)} = \begin{bmatrix} \omega_0 & \omega_0 \\ \omega^3 & \omega \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} S_{C,2,0} & S_{C,2,1} \end{bmatrix}^T,$$

where $\omega = e^{-j2\pi/2} = -1$. $S_{C,2,0} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $S_{C,2,1} = \begin{bmatrix} 1 & -1 \end{bmatrix}$. In addition, $S_{C,2,0}$ and $S_{C,2,1}$ correspond to polyphase sequences of length 2.

For $n = 4$, $S_{C,4}$ over the complex field is shown as

$$S_{C,4}^{(4)} = \begin{bmatrix} \omega_0 & \omega_0 & \omega_0 & \omega_0 \\ \omega_0 & \omega^3 & \omega & \omega^2 \\ \omega_0 & \omega & \omega^3 & \omega^2 \\ \omega_0 & \omega^2 & \omega^2 & \omega^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix},$$

where $\omega = e^{-j2\pi/4} = -j$. $S_{C,4,1} = \begin{bmatrix} \omega & \omega^3 \end{bmatrix}$ for $i = 0, 1, 2, 3$. In addition, $S_{C,4,0}, S_{C,4,1}, S_{C,4,2}$ and $S_{C,4,3}$ correspond to polyphase sequences of length 4. The Kronecker product of $S_{C,2}$ and $S_{C,4}$ generates $S_{C,8}^{(2)}$ as

$$S_{C,8}^{(2)} = S_{C,2} \otimes S_{C,4} = \begin{bmatrix} S_{C,4}^{(4)} & S_{C,4}^{(4)} & S_{C,4}^{(4)} & S_{C,4}^{(4)} \end{bmatrix} = \begin{bmatrix} S_{C,8,0} & S_{C,8,1} & S_{C,8,2} & S_{C,8,3} \end{bmatrix}^T,$$

where $S_{C,8,0} = \begin{bmatrix} \omega_0 & \omega^0 & \omega^3 & \omega^3 \\ \omega & \omega^3 & \omega_0 & \omega^2 \\ \omega^3 & \omega^3 & \omega & \omega^2 \\ \omega^2 & \omega^0 & \omega^2 & \omega^3 \end{bmatrix}$ and $S_{C,8,1} = \begin{bmatrix} \omega_0 & \omega^0 & \omega^3 & \omega^3 \\ \omega & \omega^0 & \omega^3 & \omega^3 \\ \omega^3 & \omega^3 & \omega & \omega^2 \\ \omega^2 & \omega^3 & \omega^3 & \omega^3 \end{bmatrix}$ for $i = 0, 1, 2, 3$. These sequences of length 8 construct the code tree of the polyphase OVSF codes over the complex field. The matrix $S_{C,n}$ generating the polyphase sequence is closely related to the DFT matrix.

### 3. Properties of DFT

#### 3.1 DFT over a finite field

We introduce the DFT operation over arbitrary finite fields $GF(q)$, including prime fields and extension fields. $F$ is added to a matrix or a sequence over the finite field as a
subscript in this paper. Moreover, $a$ is an element in the finite field and $n$ is a positive integer that satisfies $a^n = 1$ and $a^i \neq 1$ for $i = 1, 2, \ldots, n-1$. In the finite field, we define $V = \begin{bmatrix} v_0 & v_1 & \cdots & v_{n-1} \end{bmatrix}$ of length $n$. When referring to the notation shown in [12], the DFT operation of $V$ generates $U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{n-1} \end{bmatrix}^T$ as

$$U = S_{F,n} V,$$

where $S_{F,n}$ is defined as an $n$-point DFT matrix and $S_{F,n} = S_{F,n}^T$. In addition, $s_{F,n,i}$ is the $i$th row of $S_{F,n}$ and is defined as

$$s_{F,n,i} = \begin{bmatrix} a_0 & a^i & a^{2i} & \cdots & a^{(n-1)i} \end{bmatrix},$$

for $i = 0, 1, \ldots, n-1$. Each $s_{F,n,i}$ corresponds to a spreading code sequence of length $n$ for data demultiplexing.

We define the inverse element of $a$ as $a^{-1}$ such that $a \cdot a^{-1} = 1$ over the finite field. The $n$-point inverse DFT (IDFT) matrix is defined as

$$R_{F,n} = \begin{bmatrix} a_0 & a^0 & \cdots & a^0 \\ a_0 & a_1 & \cdots & a^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a^{(n-1)} & \cdots & a^{(n-1)2} \end{bmatrix} = \begin{bmatrix} r_{F,n,0} & r_{F,n,1} & \cdots & r_{F,n,n-1} \end{bmatrix}^T,$$

where $R_{F,n} = R_{F,n}^T$, and $r_{F,n,h}$ is the $h$th row of $R_{F,n}$ for $h = 0, 1, \ldots, n-1$ and is defined as

$$r_{F,n,h} = \begin{bmatrix} a_0 & a^{-h} & a^{-2h} & \cdots & a^{-(n-1)h} \end{bmatrix}.$$  

Each $r_{F,n,h}$ corresponds to a despreading code sequence of length $n$ for data demultiplexing.

The inner product of $s_{F,n,i}$ and $r_{F,n,h}$, which is calculated by multiplying $s_{F,n,i}$ and $r_{F,n,h}^T$ over the finite field, yields

$$s_{F,n,i} r_{F,n,h}^T = \begin{cases} n_F & (h = i), \\ 0 & \text{(otherwise)}. \end{cases}$$

Therefore, $s_{F,n,i}$ and $r_{F,n,h}$ are orthogonal to each other for $h \neq i$. In Eq. (18), $n_F$ is equal to a value obtained by adding 1 $n$ times over the finite field. In order to normalize the transform as $s_{F,n,i} r_{F,n,h}^T = 1$ for $h = i$, the definition of $r_{F,n,h}$ in Eq. (17) is replaced with

$$r_{F,n,h} = n_F^{-1} \begin{bmatrix} a_0 & a^{-h} & a^{-2h} & \cdots & a^{-(n-1)h} \end{bmatrix},$$

where the normalization factor $n_F^{-1}$ is the inverse of $n_F$ such that $n_F \cdot n_F^{-1} = 1$ over the finite field. However, in this paper, the normalization factor is not used for easy understanding.

### 3.2 Number of points for DFT

In this section, we consider the DFT over a finite field denoted as GF($q$). Then, $q$ is set to $q = p$ for a prime field, where $p$ is a prime number of $p \geq 3$. In addition, $q$ is set to $q = p^m$ for an extension field, where $p$ is a prime number of $p \geq 2$, and $m$ is an integer of $m \geq 2$. Therefore, the proposed code cannot be constructed over GF(2).

A prime factorization of $q - 1$ is represented as

$$q - 1 = p_1^{e_1} p_2^{e_2} \cdots p_i^{e_i},$$

where $p_1, p_2, \ldots, p_i$ are prime numbers that are different from each other, and $e_1, e_2, \ldots, e_i$ are positive integers. We define $d$ as the number of divisors except for one, and $d$ is given by

$$d = (e_1 + 1)(e_2 + 1) \cdots (e_i + 1) - 1.$$  

Each divisor except for the value of one corresponds to the number of points for the DFT. Next, we show some examples. $F_q$ is added to a matrix or a sequence over GF($q$) as a subscript in this paper.

### 3.3 DFT over prime fields

When $q = p = 5$, the elements of GF(5) are included in the set $\{0, 1, 2, 3, 4\}$. Let $a = 2$. Then $a^4 = 1$ and $a^i \neq 1$ for $i = 1, 2, 3$.

A four-point DFT and a two-point DFT exist over GF(5), because $q-1 = 5-1 = 4 = 2^2$. The four-point DFT matrix $S_{F,4}$ over GF(5) is given as

$$S_{F,4}^{(4)} = \begin{bmatrix} a^0 & a^0 & a^0 & a^0 \\ a^0 & a^1 & a^2 & a^3 \\ a^0 & a^2 & a^4 & a^6 \\ a^0 & a^3 & a^6 & a^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 1 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix},$$

where $s_{F,4,i}^{(4)}$ is the $i$th row of $S_{F,4}^{(4)}$ for $i = 0, 1, 2, 3$. 

![Fig. 1] Conventional binary OVSF codes.
The four-point IDFT matrix $R_{F_{4},4}^{(4)}$ is given as

$$
R_{F_{4},4}^{(4)} = \begin{bmatrix}
a^0 & a^0 & a^0 & a^0 \\
a^0 & a^{-1} & a^{-2} & a^{-3} \\
a^0 & a^{-2} & a^{-4} & a^{-6} \\
a^0 & a^{-3} & a^{-6} & a^{-9}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 4 & 2 \\
1 & 4 & 1 & 4 \\
1 & 2 & 4 & 3
\end{bmatrix},
$$

(23)

where $r_{F_{4},h}^{(4)}$ is the $h$th row of $R_{F_{4},4}^{(4)}$ for $h = 0, 1, 2, 3$.

Multiplying $s_{F_{4},4}^{(4)}$ and $r_{F_{4},4,h}^{(4)}$ over GF(5) yields

$$
s_{F_{4},4}^{(4)} r_{F_{4},4,h}^{(4)} = \begin{cases}
4 & (h = i), \\
0 & \text{(otherwise)}.
\end{cases}
$$

(24)

Next, we focus on another element of GF(5) to explain other matrix size. Let $a = 4$. Then, $a^2 = 1$ and there exists the DFT of matrix size two. The two-point DFT matrix $s_{F_{2},2}^{(2)}$ over GF(5) is written as

$$
s_{F_{2},2}^{(2)} = \begin{bmatrix}
a^0 & a^0 \\
a^0 & a^{-1}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
1 & 4
\end{bmatrix},
$$

(25)

where $s_{F_{2},2}^{(2)}$ is the $i$th row of $s_{F_{2},2}^{(2)}$ for $i = 0, 1$.

The two-point IDFT matrix $R_{F_{2},2}^{(2)}$ is written as

$$
R_{F_{2},2}^{(2)} = \begin{bmatrix}
a^0 & a^0 \\
a^0 & a^{-1}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
1 & 4
\end{bmatrix},
$$

(26)

where $r_{F_{2},2,h}^{(2)}$ is the $h$th row of $R_{F_{2},2}^{(2)}$ for $h = 0, 1$.

Multiplying $s_{F_{2},2}^{(2)}$ and $r_{F_{2},2,h}^{(2)}$ over GF(5) yields

$$
s_{F_{2},2}^{(2)} r_{F_{2},2,h}^{(2)} = \begin{cases}
2 & (h = i), \\
0 & \text{(otherwise)}.
\end{cases}
$$

(27)

### 3.4 DFT over extension fields

For $p = 2$ and $m = 8$, $q$ is equal to $p^m = 2^8$. Then, there are 256 elements of $\{0, 1, a, a^2, \ldots, a^{254}\}$ in GF($2^8$), where $a$ is a primitive root of the primitive polynomial $p(x) = x^8 + x^4 + x^3 + x^2 + 1$. In this case, $a^{255} = 1$ and $a^i \neq 1$ for $i = 1, 2, \ldots, 254$. $q - 1 = 2^8 - 1 = 255 = 3 \cdot 5 \cdot 17$, then 255 has divisors of 3, 5, 15, 17, 51, 85, and 255, except for one. Each divisor corresponds to the number of points for the DFT over GF($2^8$) [12].

The 255-point DFT over GF($2^8$) maps a vector of 255 eight-bit bytes into a vector of 255 eight-bit bytes. Let $n = 255$ and $a = \alpha$ in the $n$-point DFT, then the 255-point DFT matrix is given as

$$
s_{F_{255},255}^{(255)} = \begin{bmatrix}
a^0 & a^0 & \cdots & a^0 \\
a^0 & a^1 & \cdots & a^{a_{254}} \\
\vdots & \vdots & \ddots & \vdots \\
a^0 & a^{a_{254}} & \cdots & a^{a_{254}}
\end{bmatrix}
$$

(28)

and

$$
s_{F_{255},255}^{(255)} = \begin{bmatrix}
a^0 & a^0 & \cdots & a^0 \\
a^0 & a^{-1} & \cdots & a^{-254} \\
\vdots & \vdots & \ddots & \vdots \\
a^0 & a^{-254} & \cdots & a^{-254}
\end{bmatrix},
$$

(29)

for $i = 0, 1, \ldots, 254$.

The 255-point IDFT matrix is given as

$$
R_{F_{255},255}^{(255)} = \begin{bmatrix}
a^0 & a^0 & \cdots & a^0 \\
a^0 & a^{-1} & \cdots & a^{-254} \\
\vdots & \vdots & \ddots & \vdots \\
a^0 & a^{-254} & \cdots & a^{-254}
\end{bmatrix},
$$

(30)

where $r_{F_{255},h}^{(255)}$ is the $h$th row of $R_{F_{255},255}^{(255)}$ for $h = 0, 1, \ldots, 254$.

Multiplying $s_{F_{255},255}^{(255)}$ and $r_{F_{255},h}^{(255)}$ over GF($2^8$) yields

$$
s_{F_{255},255}^{(255)} r_{F_{255},h}^{(255)} = \begin{cases}
1 & (h = i), \\
0 & \text{(otherwise)}.
\end{cases}
$$

(32)

For $h = i$, $s_{F_{255},255}^{(255)} r_{F_{255},h}^{(255)} = 1$, because the number of points for the DFT over GF($2^8$) is odd and adding 1 an odd number of times equals 1 in the case of GF($2^m$) calculation. We define $a = a^{85}$ in GF($2^8$). Then $a^3 = 1$. The three-point DFT matrix $s_{F_{3},3}^{(3)}$ over GF($2^8$) is given as

$$
s_{F_{3},3}^{(3)} = \begin{bmatrix}
a^0 & a^0 & a^0 \\
a^0 & a^1 & a^2 \\
a^0 & a^2 & a^4
\end{bmatrix}
= \begin{bmatrix}
a^0 & a^0 & a^0 \\
a^0 & a^{85} & a^{170} \\
a^0 & a^{170} & a^{85}
\end{bmatrix},
$$

(33)

where $s_{F_{3},3}^{(3)}$ is the $i$th row of $s_{F_{3},3}^{(3)}$ for $i = 0, 1, 2$.

The three-point IDFT matrix $R_{F_{3},3}^{(3)}$ is given as

$$
R_{F_{3},3}^{(3)} = \begin{bmatrix}
a^0 & a^0 & a^0 \\
a^0 & a^{-1} & a^{-2} \\
a^0 & a^{-2} & a^{-4}
\end{bmatrix}
= \begin{bmatrix}
a^0 & a^0 & a^0 \\
a^0 & a^{85} & a^{-170} \\
a^0 & a^{-170} & a^{85}
\end{bmatrix},
$$

(34)

where $r_{F_{3},3,h}^{(3)}$ is the $h$th row of $R_{F_{3},3}^{(3)}$ for $h = 0, 1, 2$.

Multiplying $s_{F_{3},3}^{(3)}$ and $r_{F_{3},3,h}^{(3)}$ over GF($2^8$) yields

$$
s_{F_{3},3}^{(3)} r_{F_{3},3,h}^{(3)} = \begin{cases}
1 & (h = i), \\
0 & \text{(otherwise)}.
\end{cases}
$$

(35)

We define $a = a^{81}$ in GF($2^8$). Then $a^3 = 1$. The five-point DFT matrix $s_{F_{5},5}^{(5)}$ in GF($2^8$) is written as

$$
s_{F_{5},5}^{(5)} = \begin{bmatrix}
a^0 & a^0 & a^0 & a^0 & a^0 \\
a^0 & a^1 & a^2 & a^3 & a^4 \\
a^0 & a^2 & a^4 & a^6 & a^8 \\
a^0 & a^3 & a^6 & a^9 & a^{12} \\
a^0 & a^4 & a^8 & a^{12} & a^{16}
\end{bmatrix}
$$

(36)
When an extension field, where $GF(r)$, we generalize the construction of the proposed codes over finite fields. The five-point IDFT matrix $R_{F,a,s}^{(S)}$ is written as

$$R_{F,a,s}^{(S)} = \begin{bmatrix}
\alpha_0^0 & \alpha_0^0 & \alpha_0^0 & \alpha_0^0 & \alpha_0^0 \\
\alpha_0^0 & \alpha_0^{-51} & \alpha_0^{-102} & \alpha_0^{-204} & \alpha_0^{-153} \\
\alpha_0^0 & \alpha_0^{-102} & \alpha_0^{-204} & \alpha_0^{-51} & \alpha_0^{-153} \\
\alpha_0^{-204} & \alpha_0^{-153} & \alpha_0^{-51} & \alpha_0^{-102} & \alpha_0^0 \\
\alpha_0^{-102} & \alpha_0^{-204} & \alpha_0^{-51} & \alpha_0^{-153} & \alpha_0^0 
\end{bmatrix},$$

where $s_{F,a,s,j}^{(S)}$ is the $j$th row of $S_{F,a,s}^{(S)}$ for $i = 0, 1, 2, 3, 4$.

The multiplication of the $i$th row of the $n$-point DFT matrix and the $h$th column of the $n$-point IDFT matrix gives 0 over finite fields for $i \neq h$, where $i = 0, 1, \ldots, n-1$ and $h = 0, 1, \ldots, n-1$, as shown in Eq. (18). Using this fact, we propose tree-structured OVSF codes over finite fields.

### 4. Proposed OVSF codes over finite fields

The multiplication of the $i$th row of the $n$-point DFT matrix and the $h$th column of the $n$-point IDFT matrix gives 0 over finite fields for $i \neq h$, where $i = 0, 1, \ldots, n-1$ and $h = 0, 1, \ldots, n-1$, as shown in Eq. (18). Using this fact, we propose tree-structured OVSF codes over finite fields.

#### 4.1 Construction of the proposed codes

We generalize the construction of the proposed codes over $GF(q)$. Then, $q$ is set to $q = p$ for a prime field, where $p$ is a prime number of $p \geq 3$. In addition, $q$ is set to $q = p^m$ for an extension field, where $p$ is a prime number of $p \geq 2$ and $m$ is an integer of $m \geq 2$.

We assume that the number of layers is $L$ for $L \geq 2$. For $q - 1$ is factorized using prime numbers $p_1, p_2, \ldots, p_i$, for $i \geq 1$, as shown in Eq. (20), $q - 1$ has $d$ divisors, except for the divisor of value one, as shown in Eq. (21). We define these $d$ divisors as $f_1, f_2, \ldots, f_d$. The $n_1$-point DFT matrix $S_{F,a,n_1}^{(S)}$ is applied to construct the codes for $l = 1, 2, \ldots, L$. Each $n_1$ is selected from $f_1, f_2, \ldots, f_d$ arbitrarily. Therefore, there exist $d^r$ kinds of code tree. The code length in Layer 1 is defined as $k_1$ and is decided as $k_1 = \prod_{i=1}^{n_1} f_i$.

Next, we present examples. In the case of $GF(q)$ for $q = p = 7$, $q - 1$ is factorized as $q - 1 = 6 = 2 \cdot 3$, then the divisors of $q - 1$ are $f_1 = 2$, $f_2 = 3$ and $f_3 = 6$, and the number of divisors is $d = 3$. For $L = 3$, there exist $27(= d^L = 3^3)$ kinds of code tree. Table 1 shows 27 kinds of $(n_1, n_2, n_3)$, where each $n_1$ is selected from $f_1 = 2$, $f_2 = 3$ and $f_3 = 6$ for $l = 1, 2, 3$. Then Table 1 also shows 27 kinds of code lengths $(k_1, k_2, k_3)$ corresponding to $(n_1, n_2, n_3)$.

In the case of $GF(q)$ for $q = 2^s$, $q - 1$ is factorized as $q - 1 = 3 \cdot 5 \cdot 7$. Then, the number of divisors is $d = 7$, as shown in Eq. (21). For $L = 3$, there exist $343(= d^L = 7^3)$ kinds of code tree.

We show a generalized construction method for the proposed codes over $GF(q)$ of both prime and extension fields. Let $a$ be an element in $GF(q)$ which satisfies $a^{q-1} = 1$ and $a^{q-i} \neq 1$ for $i = 1, 2, \ldots, q-2$, then the $n_1$-point DFT matrix $S_{F,a,n_1}^{(S)}$ for Layer 1 is written as

$$S_{F,a,n_1}^{(S)} = \begin{bmatrix}
S_{F,a,n_1,0}^{(S)} & S_{F,a,n_1,1}^{(S)} & \cdots & S_{F,a,n_1,n_1-1}^{(S)} 
\end{bmatrix}^T,$$

where $S_{F,a,n_1,i}$ is the $i$th row of $S_{F,a,n_1}^{(S)}$ and corresponds to the code sequence of length $n_1$ in Layer 1 for $b_1 = a^{q-i}/n_1$ and $i = 0, 1, \ldots, n_1 - 1$. In addition, the $n_2$-point DFT matrix $S_{F,a,n_2}^{(S)}$ is written as

$$S_{F,a,n_2}^{(S)} = \begin{bmatrix}
S_{F,a,n_2,0}^{(S)} & S_{F,a,n_2,1}^{(S)} & \cdots & S_{F,a,n_2,n_2-1}^{(S)} 
\end{bmatrix}^T,$$

where $S_{F,a,n_2,i}$ is the $i$th row of $S_{F,a,n_2}^{(S)}$ for $b_2 = a^{q-i}/n_2$ and $i = 0, 1, \ldots, n_2 - 1$. The Kronecker product of $S_{F,a,n_1}^{(S)}$ and $S_{F,a,n_2}^{(S)}$ generates the $n_1n_2$-point matrix $S_{F,a,n_1n_2}^{(S)}$ for Layer 2 as

$$S_{F,a,n_1n_2}^{(S)} = S_{F,a,n_1}^{(S)} \otimes S_{F,a,n_2}^{(S)} = \begin{bmatrix}
S_{F,a,n_1n_2,0}^{(S)} & S_{F,a,n_1n_2,1}^{(S)} & \cdots & S_{F,a,n_1n_2,n_1n_2-1}^{(S)} 
\end{bmatrix}^T,$$

where $S_{F,a,n_1n_2,i}$ is the $i$th row of $S_{F,a,n_1n_2}^{(S)}$ and corresponds to the code sequence of length $n_1n_2$ in Layer 2 for $i = 0, 1, \ldots, n_1n_2 - 1$. Then, the $n_2$ code sequences $S_{F,a,n_1n_2,i}$ for $i = 0, 1, \ldots, n_1n_2 - 1$ are the descendants of $S_{F,a,n_1n_2,i}$ of Layer 2 for $i_1 = 0, 1, \ldots, n_1 - 1$.

### Table 1

<table>
<thead>
<tr>
<th>$(n_1, n_2, n_3)$</th>
<th>Code lengths $(k_1, k_2, k_3)$ for $L = 3$ and $GF(7)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(n_1, n_2, n_3)$</td>
<td>$(k_1, k_2, k_3)$</td>
</tr>
<tr>
<td>$(1, 2, 3)$</td>
<td>$(3, 4, 5)$</td>
</tr>
<tr>
<td>$(2, 1, 3)$</td>
<td>$(4, 5, 6)$</td>
</tr>
<tr>
<td>$(3, 1, 2)$</td>
<td>$(5, 6, 7)$</td>
</tr>
</tbody>
</table>

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0, 1, ..., \( n_1 - 1 \) in the code tree shown in Fig. 2.

For \( l = 2, 3, \ldots, L \), the Kronecker product of the \( n_1 \)-point DFT matrix \( S_{F_{x,1}}^{(n_1)} \) and the \( n_1 n_2 \cdots n_{l-1} \)-point matrix \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_{l-1})} \) generates the \( n_1 n_2 \cdots n_l \)-point matrix \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_l)} \) for Layer \( l \) as

\[
S_{F_{x,1}}^{(n_1 n_2 \cdots n_l)} = S_{F_{x,1}}^{(n_1)} \otimes S_{F_{x,1}}^{(n_1 n_2 \cdots n_{l-1})}
= \begin{bmatrix}
S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot 0)} & S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot 1)} & \cdots & S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot n_1 - 1)}
\end{bmatrix}^T,
\]

(42)

where \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot i)} \) is the \( i \)th row of \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_l)} \) and corresponds to the code sequence of length \( n_1 n_2 \cdots n_l \) in Layer \( l \) for \( i = 0, 1, \ldots, n_1 n_2 \cdots n_l - 1 \). Then, the \( n_i \) code sequences \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot i)} \) are the descendants of \( S_{F_{x,1}}^{(n_1 n_2 \cdots n_l \cdot i)} \) of Layer \( l - 1 \) for \( i = 0, 1, \ldots, n_1 n_2 \cdots n_l - 1 \) in the code tree.

The code sequences obtained from the DFT matrix over GF(\( q \)) are derived by the same procedure.

### 4.2 Examples of the proposed codes

When we design the sequences over GF(5), the Kronecker product of \( S_{F_{x,2}}^{(2)} \) and \( S_{F_{x,4}}^{(4)} \) generates \( S_{F_{x,8}}^{(4)} \) as

\[
S_{F_{x,8}}^{(4)} = S_{F_{x,2}}^{(2)} \otimes S_{F_{x,4}}^{(4)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3
1 & 4 & 1 & 4 & 1 & 4 & 1 & 4
1 & 3 & 4 & 2 & 1 & 3 & 4 & 2
1 & 1 & 1 & 4 & 4 & 4 & 4 & 4
1 & 2 & 4 & 3 & 4 & 3 & 1 & 2
1 & 4 & 1 & 4 & 4 & 1 & 4 & 1
1 & 3 & 4 & 2 & 4 & 2 & 1 & 3
\end{bmatrix}^T,
\]

(43)

where \( S_{F_{x,8}}^{(4)} \) of length 8 is the \( i \)th row of \( S_{F_{x,8}}^{(4)} \) for \( i = 0, 1, \ldots, 7 \).

The inverse matrix of \( S_{F_{x,8}}^{(4)} \) is obtained as

\[
R_{F_{x,8}}^{(4)} = R_{F_{x,2}}^{(2)} \otimes R_{F_{x,4}}^{(4)} = \begin{bmatrix}
1 & R_{F_{x,4}}^{(4)} & 1 & R_{F_{x,4}}^{(4)}
1 & R_{F_{x,4}}^{(4)} & 4 & R_{F_{x,4}}^{(4)}
\end{bmatrix}^T
= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 3 & 4 & 2 & 1 & 3 & 4 & 2
1 & 4 & 1 & 4 & 1 & 4 & 1 & 4
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3
1 & 1 & 1 & 4 & 4 & 4 & 4 & 4
1 & 2 & 4 & 3 & 4 & 3 & 4 & 3
1 & 4 & 1 & 4 & 4 & 3 & 4 & 3
1 & 2 & 4 & 3 & 4 & 3 & 1 & 2
\end{bmatrix}^T,
\]

(44)

where \( R_{F_{x,8}}^{(4)} \) of length 8 is the \( i \)th row of \( R_{F_{x,8}}^{(4)} \) for \( h = 0, 1, \ldots, 7 \).

Multiplying \( S_{F_{x,8}}^{(4)} \) and \( R_{F_{x,8}}^{(4)} \) over GF(5) yields

\[
S_{F_{x,8}}^{(4)^T} R_{F_{x,8}}^{(4)} = \begin{cases}
3 & (h = i),
0 & \text{(otherwise)}.
\end{cases}
\]

(45)

When we refer to Eq. (42), the Kronecker product of \( S_{F_{x,2}}^{(2)} \) and \( S_{F_{x,8}}^{(4)} \) generates \( S_{F_{x,16}}^{(4)} \) over GF(5) as

\[
S_{F_{x,16}}^{(4)} = S_{F_{x,2}}^{(2)} \otimes S_{F_{x,8}}^{(4)} = \begin{bmatrix}
1 & S_{F_{x,8}}^{(4)} & 1 & S_{F_{x,8}}^{(4)}
1 & S_{F_{x,8}}^{(4)} & 4 & S_{F_{x,8}}^{(4)}
\end{bmatrix}^T,
\]

(46)

\[
= \begin{bmatrix}
S_{F_{x,16,0}}^{(2)} & S_{F_{x,16,2}}^{(2)} & \cdots & S_{F_{x,16,15}}^{(2)}
\end{bmatrix}^T,
\]

where \( S_{F_{x,16,0}}^{(2)} \) of length 16 is the \( i \)th row of \( S_{F_{x,16}}^{(4)} \) for \( i = 0, 1, \ldots, 15 \).

The inverse matrix of \( S_{F_{x,16}}^{(4)} \) is derived as

\[
R_{F_{x,16}}^{(4)} = R_{F_{x,2}}^{(2)} \otimes R_{F_{x,8}}^{(4)} = \begin{bmatrix}
1 & R_{F_{x,8}}^{(4)} & 1 & R_{F_{x,8}}^{(4)}
1 & R_{F_{x,8}}^{(4)} & 4 & R_{F_{x,8}}^{(4)}
\end{bmatrix}^T,
\]

(47)

\[
= \begin{bmatrix}
R_{F_{x,16,0}}^{(2)} & R_{F_{x,16,2}}^{(2)} & \cdots & R_{F_{x,16,15}}^{(2)}
\end{bmatrix}^T,
\]

where \( R_{F_{x,16,0}}^{(2)} \) of length 16 is the \( i \)th row of \( R_{F_{x,16}}^{(4)} \) for \( h = 0, 1, \ldots, 15 \).

The obtained sequences construct the code tree of the OVSF codes depending on the DFT matrices over GF(5), as shown in Fig. 3, which is used in data multiplexing. For example, the sequence [1 2 4 3], which is the first half of \( S_{F_{x,8,5}}^{(4)} \), is not orthogonal to \( S_{F_{x,4,1}}^{(4)} = [1 3 4 2] \). Moreover, the sequence [1 2 4 3 4 1 2], which is the first half of \( S_{F_{x,8,13}}^{(4)} \), or the sequence [43 12 4 3], which is the second half of \( S_{F_{x,8,13}}^{(4)} \), is not orthogonal to \( S_{F_{x,8,5}}^{(4)} = [1 3 4 2 4 2 1] \). Therefore, if one sequence is selected as the spreading code in the system, then the sequences, which are located as its ancestors or its descendants in the code tree, cannot be selected as the spreading codes.

When we design the sequences over GF(2\(^5\)), the Kronecker product of \( S_{F_{x,5}}^{(5)} \) and \( S_{F_{x,8}}^{(8)} \) generates \( S_{F_{x,15}}^{(8)} \) as

\[
S_{F_{x,15,15}}^{(8)} = S_{F_{x,5}}^{(5)} \otimes S_{F_{x,8}}^{(8)} = \begin{bmatrix}
S_{F_{x,5}}^{(5)} & S_{F_{x,5}}^{(5)} & S_{F_{x,5}}^{(5)} & \cdots & S_{F_{x,5}}^{(5)}
S_{F_{x,8}}^{(8)} & S_{F_{x,8}}^{(8)} & S_{F_{x,8}}^{(8)} & \cdots & S_{F_{x,8}}^{(8)}
\end{bmatrix}^T,
\]

(48)

where \( S_{F_{x,15,15}}^{(8)} \) of length 15 is the \( i \)th row of \( S_{F_{x,15}}^{(8)} \) for \( i = 0, 1, \ldots, 14 \). These sequences construct the code tree of the OVSF codes over GF(2\(^5\)), as shown in Fig. 4. Since the
The proposed OVSF codes based on DFT over GF(\(q\)) obtained, because the code length on Layer \(l\) is defined as \(L_l = q^{i_l} - 1\) using prime numbers, as shown in Eq. (20). Therefore, \(d^L\) kinds of code tree exist and the various code lengths are obtained, because the code length on Layer \(l\) is decided as \(\prod_{i=1}^{L_l} n_i\). The proposed codes realize multi-rate communications with various code lengths.

The conventional OVSF code [6]-[9] also has a layered structure with \(L\) Layers for \(L \geq 2\). Combining the Hadamard matrices of points \(n_1, n_2, \ldots, n_L\) generates the code sequence of Layer \(l\) for \(l = 1, 2, \ldots, L\). Then, the code lengths of the code sequences are restricted to powers of 2.

A system configuration when the proposed codes are applied to synchronous code division multiplexing system for the downlink transmission between a transmitter at an access point and receivers for \(N\) users is illustrated in Fig. 5. At the transmitter, \(\oplus\) denotes an addition circuit in GF(q). Every element in information symbols, spread sequence symbols and multiplexed sequence symbols is in GF(q). This means that the symbols are spread and multiplexed by the calculation over GF(q). Multiplexing causes no expansion of the range for symbol values.

We show the case of \(q = 2^2\) as an example. There are four elements of \(\{0, 1, \alpha, \alpha^2\}\) in GF(\(2^2\)), where \(\alpha\) is a primitive root of the primitive polynomial \(p(x) = x^2 + x + 1\). Then, \(\alpha^3 = 1\) and \(\alpha^i \neq 1\) for \(i = 1, 2\). Since \(q - 1\) has divisor of 3, the three-point DFT exists over GF(\(2^2\)). The proposed codes of length \(k_l = 3^l\) are used as the spread sequences in Layer \(l\) for \(l \geq 1\). Multiplexing the user information symbols over GF(\(2^2\)) by using these sequences generates the multiplexed sequence symbols over GF(\(2^2\)). Both the information symbols and the multiplexed sequence symbols are elements in the set \(\{0, 1, \alpha, \alpha^2\}\), which is mapped in a one-to-one way onto the set \(\{00, 01, 10, 11\}\).

At a transmission circuit, which is denoted as Tx circuit in Fig. 5, the multiplexed sequence symbols in GF(\(2^2\)) are converted to continuous-time transmission signals fed to the channel such as a radio channel, a cable channel and an optical channel. At each receiver for user, for \(l = 0, 1, \ldots, N - 1\), signals from the channel are fed to a receiving circuit, which is denoted as Rx circuit in Fig. 5, and are converted to discrete-time symbols in GF(\(2^2\)). Demultiplexing the symbols reconstructs the information symbols in GF(\(2^2\)).

Several methods transmitting the multiplexed sequence symbols in GF(\(2^2\)) can be supposed. These symbols, which consist of two-bit patterns in \(\{00, 01, 10, 11\}\), are transmitted by on-off keying signals. And they are also transmitted by 4-phase-shift keying signals. In addition, the multiplexed sequence symbols can be error correction coded to increase error resilience of transmission signals through an erroneous channel. For example, they are transmitted by coded modulation signals combined with error correction coding of code rate 2/3 and 8-phase-shift keying mapping.

In the case of GF(\(2^2\)) as an example, we have shown that the proposed scheme has several transmission methods. The other cases will be studied in future work.

At the end of this section, we illustrate a system configuration when the conventional codes [6]-[9] are applied to synchronous code division multiplexing for the downlink transmission in Fig. 6, where \(\oplus\) denotes an addition circuit over the real or the complex field. Information symbols of
each user are spread by an assigned code sequence, and the spread sequence symbols for \( N \) users are multiplexed over the real or the complex field operation. Multiplexing causes expansion of the range for symbol values.

5. Conclusion

The present paper proposed tree-structured orthogonal spreading codes with different code lengths over finite fields including prime fields and extension fields. Combining the sequences generated by the discrete Fourier transforms over finite fields realizes various lengths of the codes. The design method for the proposed codes was shown and examples of the codes were demonstrated. The proposed scheme multiplexing symbols over finite fields would have potential of low peak-to-average power ratio (PAPR) characteristics of the transmission signal and a discussion of the PAPR is future work.

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References

Fig. 5  Downlink system configuration for the proposed codes.

Fig. 6  Downlink system configuration for the conventional codes.


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