Power Transfer Theory on Linear Passive Two-Port Systems

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SUMMARY   This paper theoretically revisits linear passive two-port systems from the viewpoint of power transfer. Instead of using the conventional S_{21} magnitude, we propose generalizing the kQ product as a figure of merit for two-port performance evaluation. We explore three examples of power transfer schemes, i.e. inductive, capacitive, and resistive channels. Starting from their voltage-current equations, the kQ formula is analytically derived for each scheme. The resultant formulas look different in appearance but are all physically consistent with $\omega M/R$, which stems from the original definition of kQ product in a primitive transformer. After comprehensively learning from the three examples, we finally extend the theory to a black-box model that represents any kind of power transfer channel. In terms of general two-port Z-parameters, useful mathematical expressions are deduced for the optimum load, input impedance, and maximum power transfer efficiency. We also supplement the theory with helpful graphics that explain how the generalized kQ behaves as a function of the circuit parameters.

key words: extended ESR, generalized kQ, maximum efficiency

1. Introduction

Power transfer is an indispensable technology for every kind of electronic device since they cannot work without supplying voltage and current. We have to deliver some energy to them, which may be done by DC cable connection or RF wireless coupling means. This paper focuses upon the channel (either cable or wireless) from the power source to where the power is needed. The channel is usually composed of an input port, an output port, and linear passive elements such as capacitors, inductors, transformers, and transmission lines. As an option, the channel may incorporate non-reciprocal components such as isolators and circulators. For some wireless applications, even transmitting and receiving antennas over a free space may be involved in the channel. In any case, the key point we exploit in this paper is that the channel can always be regarded as a *linear passive two-port system*. The prime concern of power transfer engineers is how much power we can deliver through the channel to the destination or load. As we increase the input power level, the channel heats up due to the power dissipation in the passive components. Subtracting the dissipated power, the rest of the input power is delivered to the load. The delivered power percentage out of the input power is called *power transfer efficiency*.

In the development of power transfer systems, we evaluate the designed channel structures by their efficiency. If it does not meet the given specification or goal, we need to reconsider the structure. This procedure should be repeated over and over again in order to pursue the maximum available performance under some provided constraints. The modern electromagnetic-field simulator running on a high-speed computer helps this repetitional work. The engineer just programs the software to seek the possible structure while adjusting its physical shape and dimension that finally yields the expected channel performance. The electromagnetic-field simulators usually evaluate two-port systems in S-parameters. For example, S_{21} implies the signal transfer function from port 1 to port 2 [1]. One may thus think that the above optimization could be automatically done on a simulator by maximizing the magnitude of S_{21}. Actually, RF amplifiers and filters are designed with respect to S-parameters as objective functions for their circuit optimization [2]. However, we should be aware that S-parameters are valid only when the load impedance is given, e.g. 50 ohm in most cases. For any other load impedance, the S-parameters can be translated by matrix manipulation, but we still need to know at least what impedance is to be used for the source and load before we translate the S-parameters. Power transfer engineering, in contrast, often needs to create the channel before we know the load impedance. In other words, we can adjust the load impedance afterward in accordance with the system so as to pursue as high performance as possible. Even if the load impedance is specified before we start the design of the system, we should be able to develop an impedance matching circuit to be inserted between the system and the load. Since we can adjust the matching impedance, this is equivalent to the fact that the system can be loaded with any impedance we specify. This is the essential reason why the usual S-parameter analysis is insufficient for power transfer engineering.

On this background, this paper theoretically characterizes linear passive two-port systems focusing on their power transfer performance. Instead of S-parameters, the herein presented theory is based on Z-parameters, which are invariant to source and load impedances. We first explore two fundamental examples implying inductive and capacitive channels. Deduced results are well explained in words of coupling coefficient k and quality factor Q. We next consider a channel consisting of resistors. It does not store reactive energy inside at all, and therefore Q no longer makes sense in nature. Even so, the result finds that the product of k and Q properly works for efficiency estimation. We

finally visit a black-box model that totally covers the above three examples. To make the formulas versatile while keeping their analytic continuity, we propose two comprehensive concepts i.e. *extended ESR* and *generalized kQ*. Importing them into formulation, we reach quite an elegance of maximum power transfer efficiency expression applicable to any type of linear passive two-port system.

2. Symmetrical Transformer

2.1 System Configuration

Consider a symmetrical structure composed of twin coils magnetically coupled to each other. This structure is called a 1 : 1 transformer and is also known as an inductive coupler for wireless power transfer [3]. The 1 : 1 transformer may be too primitive for use in practice but is good to begin our study with. The system is described as the circuit scheme shown in Fig. 1, where L and M stand for self and mutual inductances of the coils. They have power losses due to ohmic dissipation in their wires and possibly due to electromagnetic wave radiation from them to air. These losses are altogether represented by R inserted in series to each coil, which is usually called *equivalent series resistance* (*ESR*). The transformer is excited at port #1 by a sinusoidal voltage source with internal impedance z_s operating at angular frequency ω. Port #2 is terminated by a load having impedance $r + jx$. Keep in mind that $R > 0$ and $r > 0$ for passiveness.

2.2 Power Transfer

Voltages $[v_1, v_2]$ and currents $[i_1, i_2]$ observed at the two ports are linearly related as

$$v_1 = (R + j\omega L)i_1 - j\omega M i_2 \tag{1}$$
$$v_2 = j\omega M i_1 - (R + j\omega L)i_2 \tag{2}$$

where j stands for imaginary unit, and ω denotes angular frequency of the power source. Be careful about the negative polarity on i_2 because of the outgoing current direction at port #2. Ohm's law on complex impedance leads to another relation

$$v_2 = (r + jx)i_2 \tag{3}$$

also at port #2. When i_1 is given, the above three equations are solved as

$$v_1 = \left\{R + j\omega L + \frac{\omega^2 M^2}{R + r + j(\omega L + x)}\right\} i_1 \tag{4}$$
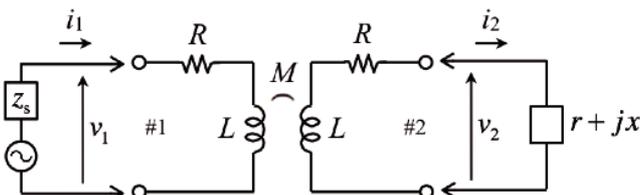

Fig. 1 Symmetric transformer circuit scheme

$$v_2 = \frac{j\omega M(r + jx)}{R + r + j(\omega L + x)} i_1 \tag{5}$$

$$i_2 = \frac{j\omega M}{R + r + j(\omega L + x)} i_1 \tag{6}$$

Putting these solutions into the definition of power transfer efficiency, we get

$$
\eta = \frac{P_2}{P_1} = \frac{\Re\{v_2 i_2^*\}}{\Re\{v_1 i_1^*\}}
$$
$$
= \frac{r\omega^2 M^2}{R(R + r)^2 + R(\omega L + x)^2 + (R + r)\omega^2 M^2} \tag{7}
$$

where $\Re$ designates the real part of a complex number. This formula tells us that efficiency η depends not only on the transformer parameters L, M, and R but also on the load impedance r and x.

2.3 Optimum Load

Looking at Eq. (7), let us seek optimum r and x that maximize η while assuming that the transformer parameters are given a priori. Assuming that variables r and x are independent to each other, we sweep x first because it appears only in the second term of the denominator. Remembering $R > 0$ and $r > 0$, the denominator is minimized when x is adjusted to its optimum

$$x_{\mathrm{opt}} = -\omega L \tag{8}$$

so that the second term becomes zero. Reflecting it back into the above denominator, we get

$$\eta = \frac{r\omega^2 M^2}{(R + r)(R^2 + \omega^2 M^2 + rR)} \tag{9}$$

We then sweep the other variable r. This is a little bit of a complicated problem because r appears not only in the denominator but also in the numerator. An algebraic operation rewrites Eq. (9) into

$$\eta = \frac{\omega^2 M^2}{R\left(\sqrt{r} - \dfrac{\sqrt{R^2 + \omega^2 M^2}}{\sqrt{r}}\right)^2 + \left(R + \sqrt{R^2 + \omega^2 M^2}\right)^2} \tag{10}$$

Remembering $R > 0$ and $r > 0$ again, all the three terms are non-negative. The last variable r stays only in the first term of denominator while other terms are all constant. The variable term is minimized when r is adjusted to its optimum

$$r_{\mathrm{opt}} = \sqrt{R^2 + \omega^2 M^2} \tag{11}$$

Reflecting it back into the above denominator again, η is finally maximized as

$$\eta_{\mathrm{max}} = \frac{\omega^2 M^2}{\left(R + \sqrt{R^2 + \omega^2 M^2}\right)^2} \tag{12}$$
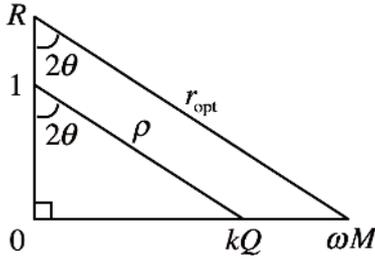
Fig. 2 Two similar right triangles to define ρ and 2θ

2.4 kQ Product

From a physical knowledge on transformers and lossy inductors, we learn

$$k = \frac{M}{L}, \quad Q = \frac{\omega L}{R} \qquad \therefore \ kQ = \frac{\omega M}{R} \tag{13}$$

where k and Q stand for coupling coefficient and quality factor. Substituting Eq. (13) into (12), we obtain

$$\eta_{\max} = \frac{k^2 Q^2}{\left(1 + \sqrt{1 + k^2 Q^2}\right)^2} \tag{14}$$

What is quite significant is that $\eta_{\max}$ is expressed in terms of only one parameter, namely kQ *product*. However, the formula still looks complicated and not so straightforward to see how it behaves as a function of kQ. To provide clarity with this equation, we define

$$\rho = \sqrt{1 + k^2 Q^2} \tag{15}$$

as a convenient substitute. Interpreting Eqs. (11) and (15) in geometry, ρ and r_{opt} imply the parallel hypotenuses of two similar right triangles displayed in Fig. 2. The above defined ρ enables us to rewrite Eq. (14) as

$$\eta_{\max} = \frac{\rho - 1}{\rho + 1} \tag{16}$$

This is one of the most prominent outcomes of this work. Also, for quick inverse verification, we append

$$\rho = \frac{1 + \eta_{\max}}{1 - \eta_{\max}} \tag{17}$$

$$kQ = \sqrt{\rho^2 - 1} \tag{18}$$

Even though we started from a transformer [3], it is worth noting that Eqs. (14) through (18) are not limited to this particular scheme but also valid for various kinds of power transfer systems [4]–[6].

2.5 Angular Expression

As an alternative way for the mathematical expression, we define 2θ as designated in Fig. 2. This graphic finds trigonometrical relations

$$kQ = \tan 2\theta \tag{19}$$

$$\rho = \sec 2\theta \tag{20}$$

Applying them to Eq. (6), the current ratio under the optimum load condition specified by Eqs. (8) and (11) translates into

$$\frac{i_2}{i_1} = \frac{j\omega M}{R + r_{\mathrm{opt}} + j(\omega L + x_{\mathrm{opt}})} = \frac{j\omega M}{R + \sqrt{R^2 + \omega^2 M^2}}$$

$$= \frac{jkQ}{1 + \rho} = \frac{j \sin 2\theta}{1 + \cos 2\theta} = j \tan \theta \tag{21}$$

The tangent does not exceed unity for θ ranging from 0 to 45 degrees. This is geometrically explained by the right triangle staying in shape. Also note that the pure imaginary number means the output current running 90 degrees faster (or 270 degrees slower) than the input in phase. In a similar manner, the maximum efficiency translates into

$$\eta_{\max} = \frac{\sec 2\theta - 1}{\sec 2\theta + 1} = \tan^2 \theta \tag{22}$$

from Eq. (16). It is physically persuasive to be exactly equal to the modulus square of the current ratio.

2.6 Input Impedance

Lastly we concern ourselves with the input impedance of the transformer. It can be derived as

$$z_{\mathrm{in}} = \frac{v_1}{i_1} = R + j\omega L + \frac{\omega^2 M^2}{R + r + j(\omega L + x)} \tag{23}$$

from Eq. (4). Especially when we employ the optimum load given in Eqs. (8) and (11), the above impedance calculates

$$z_{\mathrm{in}} = R + j\omega L + \frac{\omega^2 M^2}{R + \sqrt{R^2 + \omega^2 M^2}}$$

$$= \sqrt{R^2 + \omega^2 M^2} + j\omega L \tag{24}$$

Once z_{in} is known, we can choose the power source that matches this transformer. To exhaustively squeeze the source up to its last piece of power, we should impose the conjugate matching condition on the impedance. From Eqs. (8), (11), and (24), we find the optimum source impedance

$$z_{\mathrm{s}} = z_{\mathrm{in}}^* = \sqrt{R^2 + \omega^2 M^2} - j\omega L$$

$$= r_{\mathrm{opt}} + jx_{\mathrm{opt}} \tag{25}$$

This result leads us to conclude that the source works best when it is designed to have the same impedance as the optimum load impedance [7], [8].

Note that actual transformers do not always have symmetrical structures. They also suffer from some parasitic effects such as stray capacity between wire turns of the coil [9]. It is therefore unrigorous to model them with only primitive elements like L and M. In that case, the formulas derived in this chapter must be somewhat modified by reflecting practical effects. To address these problems, we

have to wait for the general two-port modeling and formulation to appear in chapter 5.

3. π-Shape Capacitive Network

3.1 System Configuration

The second example we consider consists of three capacitors as shown in Fig. 3. Conductance G physically implies the presence of possible but usually undesired current leakage between the electrodes at each port. The load is symbolized by g and b in the admittance domain. Taking these parameters into account, the port voltages and currents behave so as to satisfy three circuit equations

$$i_1 = \{G + j\omega(C_1 + C_2)\}v_1 - j\omega C_2 v_2 \tag{26}$$
$$i_2 = j\omega C_2 v_1 - \{G + j\omega(C_1 + C_2)\}v_2 \tag{27}$$
$$i_2 = (g + jb)v_2 \tag{28}$$

We find that they have exactly the same fashions as Eqs. (1) to (3) except for voltage-current replacement at each port, i.e. $v_1 \Leftrightarrow i_1$ and $v_2 \Leftrightarrow i_2$.

3.2 Duality Theorem

Thanks to the duality of circuit impedance and admittance, we do not have to repeat the formulation done in the previous chapter. All we need is just to replace the parameters as

$$L \Rightarrow C_1 + C_2, \quad M \Rightarrow C_2, \quad R \Rightarrow G$$
$$r \Rightarrow g, \quad x \Rightarrow b \tag{29}$$

in the formulas that we assumed and deduced for the symmetrical transformer. For example, we describe $G > 0$ and $g > 0$ instead of $R > 0$ and $r > 0$.

3.3 Optimum Load

Applying Eq. (29) to Eqs. (8) and (11), we find

$$b_{\text{opt}} = -\omega(C_1 + C_2)$$
$$g_{\text{opt}} = \sqrt{G^2 + \omega^2 C_2^2} \tag{30}$$

for the optimum load susceptance and conductance.

3.4 Maximum Efficiency

When the above optimum load is employed, the voltage

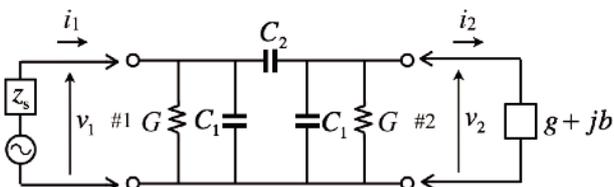transfer ratio and kQ product result in

$$\frac{v_2}{v_1} = j\tan\theta \tag{31}$$

$$k = \frac{C_2}{C_1 + C_2}, \qquad Q = \frac{\omega(C_1 + C_2)}{G}$$
$$\therefore \ kQ = \frac{\omega C_2}{G} = \tan 2\theta \tag{32}$$

We consequently reach the triangle hypotenuse length, maximum power transfer efficiency, and optimum source impedance, respectively as

$$\rho = \sqrt{1 + k^2 Q^2} = \sec 2\theta \tag{33}$$

$$\eta_{\max} = \frac{\omega^2 C_2^2}{\left(G + \sqrt{G^2 + \omega^2 C_2^2}\right)^2}$$
$$= \frac{\rho - 1}{\rho + 1} = \tan^2\theta \tag{34}$$

$$z_{\text{s}} = z_{\text{in}}^* = \frac{1}{g_{\text{opt}} + jb_{\text{opt}}} \tag{35}$$

4. T-Shape Resistive Network

4.1 System Configuration

The third example we consider consists of three resistors as shown in Fig. 4. Unlike inductors and capacitors, resistors are free of reactive energy [10]. We thus characterize the system by employing real-number voltages and currents. A dc battery is assumed for the power source. The load impedance does not need reactance either. This simple configuration gives us the essential sense of power transfer.

4.2 Power Transfer

From the circuit diagram given in Fig. 4, Kirchhoff's voltage law leads to three equations

$$v_1 = (R_1 + R_2)i_1 - R_2 i_2 \tag{36}$$
$$v_2 = R_2 i_1 - (R_2 + R_3)i_2 \tag{37}$$
$$v_2 = r i_2 \tag{38}$$

Solving them for the voltages and currents, and then putting the results into the dc power transfer efficiency, we obtain

$$\eta = \frac{P_2}{P_1} = \frac{v_2 i_2}{v_1 i_1} = \frac{(AC - B)r}{(Ar + B)(r + C)} \tag{39}$$
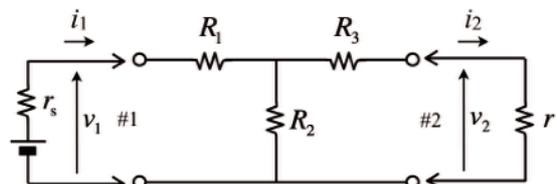
where A, B, and C are locally defined polynomials

Fig. 3　π-shape capacitive network

Fig. 4　T-shape resistive network

$$A = R_1 + R_2 \tag{40}$$

$$B = R_1R_2 + R_1R_3 + R_2R_3 \tag{41}$$

$$C = R_2 + R_3 \tag{42}$$

for appearance neatness.

4.3 Optimum Load

To find the optimum load that maximizes the efficiency, we rewrite Eq. (39) as

$$\eta = \frac{AC - B}{Ar + \dfrac{BC}{r} + AC + B}$$

$$= \frac{AC - B}{\left(\sqrt{Ar} - \sqrt{\dfrac{BC}{r}}\right)^2 + \left(\sqrt{AC} + \sqrt{B}\right)^2} \tag{43}$$

The variable r appears only in the first term of the denominator, while all other terms are constant. The first term is minimized by tuning r in

$$r_{\text{opt}} = \sqrt{\frac{BC}{A}} \tag{44}$$

Putting it back into Eq. (43), the efficiency reaches its peak

$$\eta_{\text{max}} = \frac{AC - B}{\left(\sqrt{AC} + \sqrt{B}\right)^2} = \frac{\sqrt{AC} - \sqrt{B}}{\sqrt{AC} + \sqrt{B}} \tag{45}$$

Subsequently, the input resistance becomes

$$r_{\text{in}} = \sqrt{\frac{AB}{C}} \tag{46}$$

when we employ the optimum load.

4.4 Equivalent kQ Product

Recalling the physical meaning of Q, it does not basically work for resistive systems. This is because resistors have neither reactive stored energy nor any frequency slope at all [4], [5]. Even if it is the case, we can metaphysically exploit the kQ formula derived in chapter 2. This is actually done by redefining

$$\rho = \sqrt{\frac{AC}{B}} \tag{47}$$

$$kQ = \frac{R_2}{\sqrt{R_1R_2 + R_1R_3 + R_2R_3}} \tag{48}$$

so that Eq. (45) becomes consistent with Eqs. (14) to (18). Comparing this kQ with its original definition introduced in Eq. (13), we find out the replacement

$$\omega M \Rightarrow R_2 \tag{49}$$

$$R \Rightarrow \sqrt{R_1R_2 + R_1R_3 + R_2R_3} \tag{50}$$

These two quantities could therefore be called *mutual resistance* and *extended ESR*, respectively.

5. General Black Box

5.1 System Configuration

For the sake of establishing a unified theory that covers the three examples discussed in previous chapters, consider a black-box model shown in Fig. 5. The box may contain any kind of linear passive circuit component, which may be lumped- or distributed-constant elements, planar circuits, waveguides, or even aerial structures [11]. The internal topology of the box does not matter, but we just need its two-port parameters to construct the theory.

5.2 Circuit Equations

Even without knowing the box's internal information, we can just observe its Z-parameters

$$Z_{11} = R_{11} + jX_{11}, \quad Z_{12} = R_{12} + jX_{12}$$
$$Z_{21} = R_{21} + jX_{21}, \quad Z_{22} = R_{22} + jX_{22} \tag{51}$$

by looking into the box from its two ports. Once we have these parameters, the voltage-current relations are formulated as

$$v_1 = Z_{11}i_1 - Z_{12}i_2 \tag{52}$$

$$v_2 = Z_{21}i_1 - Z_{22}i_2 \tag{53}$$

along with the loading condition

$$v_2 = zi_2 = (r + jx)i_2 \tag{54}$$

at port #2.

5.3 Power Transfer

Solving the above three equations for the voltages and currents, and substituting them into the power transfer efficiency, we obtain

$$\eta = \frac{P_2}{P_1} = \frac{\Re\{v_2 i_2^*\}}{\Re\{v_1 i_1^*\}} = \frac{r\left|\dfrac{Z_{21}}{z + Z_{22}}\right|^2}{\Re\left\{Z_{11} - \dfrac{Z_{12}Z_{21}}{z + Z_{22}}\right\}}$$

$$= \frac{|Z_{21}|^2}{R_{11}r + \dfrac{A}{r} + \Sigma + \Delta} \tag{55}$$
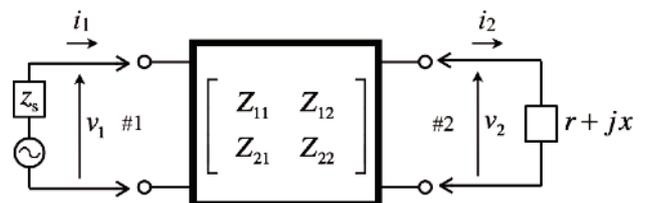

Fig. 5 Two-port black box for general formulation

where the local polynomials

$$\Sigma = R_{11}R_{22} + X_{12}X_{21} \tag{56}$$

$$\Delta = R_{11}R_{22} - R_{12}R_{21} \tag{57}$$

$$A = R_{11}x^2 + Bx + C \tag{58}$$

$$B = 2R_{11}X_{22} - R_{12}X_{21} - R_{21}X_{12} \tag{59}$$

$$C = (R_{11}R_{22} - R_{12}R_{21} + X_{12}X_{21})R_{22}$$
$$\quad + (R_{11}X_{22} - R_{12}R_{21} - R_{21}X_{12})X_{22} \tag{60}$$

are defined for arithmetic convenience. Note that, for any passive system, none of the four terms in the denominator of Eq. (55) can be negative. Particularly for the second term, we assume that $A \geq 0$ for any x. This is proved by contradiction as follows. If A were negative for some x, the denominator could be nullified for that x with some positive r, and thus the efficiency would exceed unity, which can never be true in practice.

5.4   Optimum Load

In the same way as we have done in Sect. 2.3, let us seek optimum r and x that maximize η while assuming that the box's Z-parameters are given a priori. Also assuming that variables r and x are independent to each other, we sweep x first because it appears only in the term of A. Recalling that $r > 0$, the right-hand side of Eq. (55) is maximized when we adjust x so as to minimize A. Since Eq. (58) can be rewritten as

$$A = R_{11}\left(x + \frac{B}{2R_{11}}\right)^2 - \frac{B^2}{4R_{11}} + C \tag{61}$$

the optimum x is found as

$$x_{\mathrm{opt}} = -\frac{B}{2R_{11}} = \frac{R_{12}X_{21} + R_{21}X_{12}}{2R_{11}} - X_{22} \tag{62}$$

Putting it back into Eq. (61), A reaches its minimum

$$A_{\min} = -\frac{B^2}{4R_{11}} + C = \frac{\Sigma\Delta - \Theta^2}{R_{11}} \tag{63}$$

where

$$\Theta = \frac{1}{2}(R_{12}X_{21} - R_{21}X_{12}) \tag{64}$$

is called *impedance exchange term* in this paper. When A is minimized, Eq. (55) becomes

$$\eta = \frac{|Z_{21}|^2}{R_{11}r + \dfrac{A_{\min}}{r} + \Sigma + \Delta} \tag{65}$$

Being finished with x now, we next sweep r to further minimize the denominator. It is transformed into a complete square for r as

$$\eta = \frac{|Z_{21}|^2}{\left(\sqrt{R_{11}r} - \sqrt{\dfrac{A_{\min}}{r}}\right)^2 + 2\sqrt{R_{11}A_{\min}} + \Sigma + \Delta} \tag{66}$$

From the first term of the denominator, the optimum solution for r is found as

$$r_{\mathrm{opt}} = \sqrt{\frac{A_{\min}}{R_{11}}} = \frac{\sqrt{\Sigma\Delta - \Theta^2}}{R_{11}} \tag{67}$$

Putting it back into Eq. (66) and with help of Eq. (63), we obtain the maximum efficiency

$$\eta_{\max} = \frac{|Z_{21}|^2}{\Sigma + \Delta + 2\sqrt{\Sigma\Delta - \Theta^2}} \tag{68}$$

Putting Eqs. (62) and (67) into Eqs. (52) to (54), and then solving them for v_1/i_1, we find the input impedance

$$z_{\mathrm{in}} = r_{\mathrm{in}} + jx_{\mathrm{in}}$$
$$\begin{cases} r_{\mathrm{in}} = \dfrac{\sqrt{\Sigma\Delta - \Theta^2}}{R_{22}} \\[2ex] x_{\mathrm{in}} = X_{11} - \dfrac{R_{12}X_{21} + R_{21}X_{12}}{2R_{22}} \end{cases} \tag{69}$$

under the optimum load condition.

5.5   Reciprocity

The black box being considered in this chapter consists of linear passive elements. In addition to this assumption, practical power transfer systems usually exhibit *reciprocity* between the two ports as well. This is because most of the linear passive elements have this property except for special-purpose components such as isolators and circulators [12].

In words of Z-parameters given by Eq. (51), reciprocity is defined as

$$\begin{cases} R_{12} = R_{21} \\ X_{12} = X_{21} \end{cases} \tag{70}$$

This brings efficacious simplicity to every equation derived in the previous section. Saliently, Eq. (64) goes to zero by reciprocity. Accordingly, the impedance exchange term Θ vanishes from all equations after that. Among them, we pick out some significant formulas below.

The optimum load impedance to maximize the power transfer efficiency results in

$$z_{\mathrm{opt}} = r_{\mathrm{opt}} + jx_{\mathrm{opt}}$$
$$\begin{cases} r_{\mathrm{opt}} = \dfrac{\sqrt{\Sigma\Delta}}{R_{11}} \\[2ex] x_{\mathrm{opt}} = \dfrac{R_{12}X_{21}}{R_{11}} - X_{22} \end{cases} \tag{71}$$

Under this optimum loading condition, the black box exhibits its input impedance

$$z_{\mathrm{in}} = r_{\mathrm{in}} + jx_{\mathrm{in}}$$
$$\begin{cases} r_{\mathrm{in}} = \dfrac{\sqrt{\Sigma\Delta}}{R_{22}} \\[2ex] x_{\mathrm{in}} = X_{11} - \dfrac{R_{12}X_{21}}{R_{22}} \end{cases} \tag{72}$$
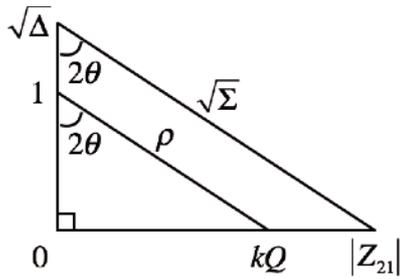
and achieves its maximum efficiency

Fig. 6 Graphical visualization of Eqs. (74) and (76)

$$\eta_{\max} = \frac{\sqrt{\Sigma} - \sqrt{\Delta}}{\sqrt{\Sigma} + \sqrt{\Delta}} \tag{73}$$

This formula is so elegant that everyone can keep it in mind with ease. All we need to do is just recall Σ and Δ from Eqs. (56) and (57).

To comprehend Eq. (73) from the viewpoint of physics, we exploit the kQ concept again that appeared for the transformer scheme in chapter 2. For this purpose, we metaphysically redefine

$$kQ = \frac{|Z_{21}|}{\sqrt{\Delta}} \tag{74}$$

Employing this definition, Eq. (73) is consistently represented by

$$\eta_{\max} = \frac{\rho - 1}{\rho + 1} \tag{75}$$

where

$$\rho = \sqrt{1 + (kQ)^2} \tag{76}$$

They are exactly identical to Eqs. (15) and (16). Namely, Eq. (74) should be called *generalized kQ*.

To provide a helpful supplement to our comprehension, Eqs. (74) and (76) are visualized into graphics as shown in Fig. 6. The two similar right triangles notify us of trigonometrical relations

$$kQ = \tan 2\theta = \rho \sin 2\theta \tag{77}$$

$$\rho = \sec 2\theta = \sqrt{\frac{\Sigma}{\Delta}} \tag{78}$$

We find that they are consistent extensions from Eqs. (19) and (20) to the general black box under the reciprocity assumption. One more instructive visualization is shown in Fig. 7 from Eqs. (75) and (77). The two right triangles cooperatively enable us to see how $\eta_{\max}$ goes up as kQ increases.

Note that the generalized kQ is a new concept, not just the product of conventional k and Q. In other words, Eq. (74) covers not only Eqs. (13) and (32) but also is useful for the complicated schemes where it cannot be factorized into k and Q. We thus should respect the generalized kQ as a unified figure of merit for power transfer engineering [5].

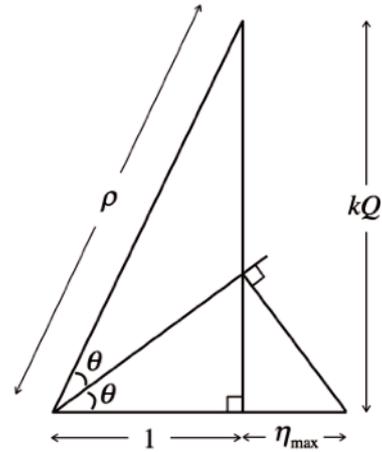To get even more familiar with the generalized kQ, we compare Eq. (74) with the original kQ product introduced

Fig. 7 Graphical visualization of Eqs. (75) and (77)

for the transformer in Eq. (13). Looking at the two formulas with a focus on each numerator and denominator separately, we notice the replacement

$$\omega M \Rightarrow \sqrt{R_{21}^2 + X_{21}^2} \tag{79}$$

$$R \Rightarrow \sqrt{R_{11}R_{22} - R_{12}R_{21}} \tag{80}$$

in the same way as taken for Eqs. (49) and (50). We hereby find out that the mutual inductance is just an instance of *trans-impedance magnitude*. The equivalent series resistance (*ESR*) of a coil is just a dimensional reduction from a two-port resistance matrix. Conversely, the above square root should be called *extended ESR*.

6. Conclusion

Linear passive two-port systems were theoretically revisited from the aspect of power transfer. As a main concern in power engineering, the maximum power transfer efficiency was formulated in terms of two-port Z-parameters. On the journey to this formulation, we presented the concepts of extended *ESR* and generalized *kQ*. Featuring these new concepts, the established theory covers transformers, inductive, capacitive, and resistive networks, and any other linear passive two-port system. Being invariant to loading conditions, the generalized *kQ* works as a versatile figure of merit for use in the design and evaluation of power transfer systems. This also enables us to impartially compare the designed system with its alternatives even before knowing their loading conditions. Once a high-kQ system is obtained, we can choose the optimum power source and load, or design the optimum impedance matching circuits to connect the system with specified source and load. For this purpose, we also theoretically deduced the optimum load and input impedance formulas. Since the formulas created in this paper are mathematically explicit and thus ready to use in computer-aided design and structure optimization, they will significantly contribute to sophisticated power transfer engineering and future system developments

involving distributed-constant elements such as transmission lines, spatial propagation modes between antennas, and networks with multiple sources and loads [5], [6], [11].

Acknowledgments

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